# Stochastic Optimal Control in Infinite Dimensions

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Minimize the cost functional

$$J(u) := \frac{1}{2}\mathbb{E}\left[\nu \int_0^1 u_s^2 \mathrm{d}s + (x_1^u)^2\right]$$

over $u \in L^2_{\mathcal{F}}([0,1]\times\Omega)$ subject to the state equation

$$\begin{cases} \mathrm{d}x_s^u = [\theta(\mu - x_s^u) + u_s]\,\mathrm{d}s + \sigma \mathrm{d}W_s \quad \text{on } [0,1]\times\mathbb{R} \\ x_0^u = x \in \mathbb{R}. \end{cases}$$

Parameters: $\nu = 1/100$, $\theta = 3$, $\mu = 1$, $\sigma = 1$, $x = 0$.

Figure: Uncontrolled Trajectories

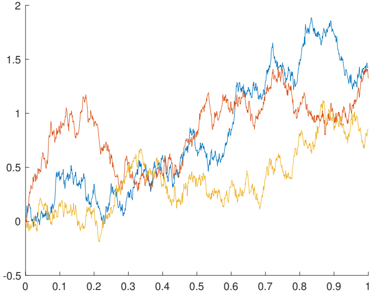
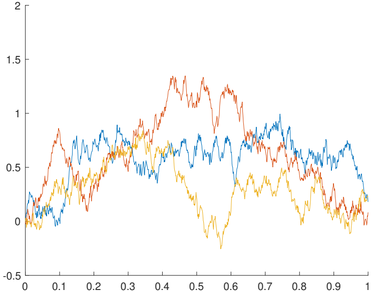



Figure: Controlled Trajectories



# Finite-Dimensional Setting

Minimize the cost functional

$$J(u) := \mathbb{E}\left[\int_0^T l(x_s^u, u_s)\mathrm{d}s + h(x_T^u)\right]$$

over $u \in L^2_{\mathcal{F}}([0,T] \times \Omega)$ subject to the state equation

$$\begin{cases} \mathrm{d}x_s^u = b(x_s^u, u_s)\mathrm{d}s + \sigma(x_s^u, u_s)\mathrm{d}W_s & \text{on } [0,T] \times \mathbb{R} \\ x_t^u = x \in \mathbb{R}, \end{cases}$$

where

- $l : \mathbb{R} \times U \to \mathbb{R}$ are running costs,
- $h : \mathbb{R} \to \mathbb{R}$ are terminal costs,
- $b$ and $\sigma$ are drift and diffusion coefficient, respectively,
- $(W_s)_{s\in[0,T]}$ is a Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\in[0,T]}, \mathbb{P})$.

# Necessary Optimality Condition

Derive necessary optimality condition:

$$\langle \nabla J(u), v \rangle = \partial_v J(u) = \mathbb{E}\left[\int_0^T l_x(x_s^u, u_s)\partial_v x_s^u + l_u(x_s^u, u_s) v_s \mathrm{d}s + h_x(x_T^u)\partial_v x_T^u\right] \overset{!}{=} 0.$$

We want to separate $v$ in order to identify the gradient.
First step: $y_t^v := \partial_v x_t^u$ solves the SDE

$$\begin{cases} \mathrm{d}y_t^v = \left[b_x(x_t^u, u_t) y_t^v + b_u(x_t^u, u_t) v_t\right] \mathrm{d}t + \left[\sigma_x(x_t^u, u_t) y_t^v + \sigma_u(x_t^u, u_t) v_t\right] \mathrm{d}W_t \\ y_0^v = 0. \end{cases}$$

Second step: In the deterministic case, one introduces the adjoint state

$$\begin{cases} \mathrm{d}p_t = -\left[b_x(x_t^u, u_t) p_t + l_x(x_t^u, u_t)\right] \mathrm{d}t \\ p_T = h_x(x_T^u). \end{cases}$$

Random terminal condition $\rightsquigarrow$ solution is not adapted to given filtration.

# Backward Stochastic Differential Equations

Let $(W_t)_{t\geq 0}$ be a Brownian motion and $(\mathcal{F}_t)_{t\geq 0}$ its natural filtration. Consider the BSDE

$$\begin{cases} \mathrm{d}p_t = 0 \\ p_T = \xi, \end{cases}$$

where $\xi$ is $\mathcal{F}_T$-measurable. Natural candidate: $p_t = \mathbb{E}[\xi|\mathcal{F}_t]$.

⚠ Adapted solution doesn't exist!

Martingale representation theorem:

$$\mathbb{E}[\xi|\mathcal{F}_t] = \mathbb{E}[\xi] + \int_0^t q_s \mathrm{d}W_s.$$

Restate original problem: Find a pair of processes $(p, q)$ satifying

$$\begin{cases} \mathrm{d}p_t = q_t \mathrm{d}W_t \\ p_T = \xi. \end{cases}$$

# Necessary Optimality Condition Continued

Introduce adjoint state

$$\begin{cases} \mathrm{d}p_t = -\left[b_x(x_t^u, u_t)p_t + l_x(x_t^u, u_t) + \sigma_x(x_t^u, u_t)q_t\right]\mathrm{d}t + q_t \mathrm{d}W_t \\ p_T = h_x(x_T^u). \end{cases}$$

Adjoint state property:

$$\begin{aligned} \mathbb{E}\left[p_T y_T^v\right] &= \mathbb{E}\left[\int_0^T p_t \mathrm{d}y_t^v + \int_0^T y_t^v \mathrm{d}p_t + \langle p, y^v\rangle_T\right] \\ &= \mathbb{E}\left[\int_0^T b_u(x_t^u, u_t)p_t v_t + \sigma_u(x_t^u, u_t)q_t v_t - l_x(x_t^u, u_t)y_t^v \mathrm{d}t\right]. \end{aligned}$$

Hence,

$$\begin{aligned} \langle \nabla J(u), v\rangle = \partial_v J(u) &= \mathbb{E}\left[\int_0^T \left(l_u(x_t^u, u_t) + b_u(x_t^u, u_t)p_t + \sigma_u(x_t^u, u_t)q_t\right) v_t \mathrm{d}t\right] \\ &= \langle l_u(x^u, u) + b_u(x^u, u)p + \sigma_u(x^u, u)q, v\rangle = 0 \end{aligned}$$

for all $v$. Necessary condition:

$$\nabla J(u) = l_u(x^u, u) + b_u(x^u, u)p + \sigma_u(x^u, u)q = 0.$$

# Pontryagin Maximum Principle

Minimize the cost functional

$$J(u) := \mathbb{E}\left[\int_0^T l(x_s^u, u_s)\mathrm{d}s + h(x_T^u)\right]$$

over $u \in L^2_{\mathcal{F}}([0,T] \times \Omega; U)$ subject to the state equation

$$\begin{cases} \mathrm{d}x_s^u = b(x_s^u, u_s)\mathrm{d}s + \sigma(x_s^u, u_s)\mathrm{d}W_s \quad \text{on } [0,T] \times \mathbb{R}^d \\ x_t^u = x \in \mathbb{R}^d. \end{cases}$$

If $U \subset \mathbb{R}^n$ is non-convex, we need to introduce a second adjoint state

$$(P, Q) \in L^2_{\mathcal{F}}([0,T] \times \Omega; \mathbb{R}^{d \times d}) \times L^2_{\mathcal{F}}([0,T] \times \Omega; \mathbb{R}^{d \times d})^k.$$

Pontryagin Maximum Principle: Let $(\bar{x}, \bar{u})$ be optimal. Then

$$\begin{aligned} &\mathcal{H}(\bar{x}_s, \bar{u}_s, p_s, P_s) + \operatorname{tr}\left(\sigma(\bar{x}_s, \bar{u}_s)^* \left[q_s - P_s \sigma(\bar{x}_s, \bar{u}_s)\right]\right) \\ &\qquad = \inf_{u \in U} \mathcal{H}(\bar{x}_s, u, p_s, P_s) + \operatorname{tr}\left(\sigma(\bar{x}_s, u)^* \left[q_s - P_s \sigma(\bar{x}_s, \bar{u}_s)\right]\right), \end{aligned}$$

where

$$\mathcal{H}(x, u, p, P) := l(x, u) + \langle p, b(x, u)\rangle + \frac{1}{2}\operatorname{tr}\left(\sigma(x, u)^* P \sigma(x, u)\right).$$

# Dynamic Programming

Minimize

$$J(t,x;u) := \mathbb{E}\left[\int_t^T l(x_s^u, u_s)\mathrm{d}s + h(x_T^u)\right]$$

over $u \in \mathcal{U} := L^2_{\mathcal{F}}(t, T; U)$ subject to

$$\begin{cases} \mathrm{d}x_s^u = b(x_s^u, u_s)\mathrm{d}s + \sigma(x_s^u, u_s)\mathrm{d}W_s & \text{on } [t, T] \times \mathbb{R} \\ x_t^u = x \in \mathbb{R}. \end{cases}$$

Define value function

$$V(t,x) := \inf_{u \in \mathcal{U}} J(t,x;u).$$

Satisfies dynamic programming principle

$$V(t,x) = \inf_{u \in \mathcal{U}} \mathbb{E}\left[\int_t^\tau l(x_s^u, u_s)\mathrm{d}s + V(\tau, x_\tau^u)\right], \qquad \forall \tau \in [t, T].$$

# Hamilton-Jacobi-Bellman Equation

By the dynamic programming principle and Itô's formula, we have

$$
\begin{aligned}
0 &= \inf_{u \in \mathcal{U}} \mathbb{E}\left[\int_t^\tau l(x_s^u, u_s)\mathrm{d}s + V(\tau, x_\tau^u) - V(t,x)\right] \\
&= \inf_{u \in \mathcal{U}} \mathbb{E}\left[\int_t^\tau l(x_s^u, u_s) + V_t(s, x_s^u) + V_x(s, x_s^u) b(x_s^u, u_s) + \frac{1}{2} V_{xx}(s, x_s^u)\sigma^2(x_s^u, u_s)\mathrm{d}s\right].
\end{aligned}
$$

Dividing by $\tau - t$ and sending $\tau \downarrow t$ yields

$$
\begin{cases}
V_t(t,x) + \inf_{u \in U}\left\{l(x,u) + V_x(t,x)b(x,u) + \frac{1}{2}V_{xx}(t,x)\sigma^2(x,u)\right\} = 0 \\
V(T,x) = h(x).
\end{cases}
$$

Fully nonlinear PDE. Solution theory (viscosity solutions) developed in 1980s by Crandall, Evans, Lions and others. Solution is merely a continuous function.

# Optimal Synthesis

Verification Theorem: If

$$\bar{u}_s \in \arg\min \left\{ l(\bar{x}_s, u) + \langle V_x(s, \bar{x}_s), b(\bar{x}_s, u) \rangle + \frac{1}{2} V_{xx}(s, \bar{x}_s) \sigma^2(\bar{x}_s, u) \right\},$$

then $\bar{u}$ is optimal.
Indeed, by Itô's formula, we have

$$\begin{aligned} &\mathbb{E}\left[V(T, \bar{x}_T)\right] \\ &= V(t,x) + \mathbb{E}\left[\int_t^T V_t(s, \bar{x}_s) + V_x(s, \bar{x}_s) b(\bar{x}_s, \bar{u}_s) + \frac{1}{2} V_{xx}(s, \bar{x}_s) \sigma^2(\bar{x}_s, \bar{u}_s) \mathrm{d}s\right]. \end{aligned}$$

Hence,

$$\begin{aligned} V(t,x) &= \mathbb{E}\left[h(\bar{x}_T) + \int_t^T l(\bar{x}_s, \bar{u}_s) \mathrm{d}s\right] \\ &- \mathbb{E}\left[\int_t^T \underbrace{V_t(s, \bar{x}_s) + l(\bar{x}_s, \bar{u}_s) + V_x(s, \bar{x}_s) b(\bar{x}_s, \bar{u}_s) + \frac{1}{2} V_{xx}(s, \bar{x}_s) \sigma^2(\bar{x}_s, \bar{u}_s)}_{=0} \mathrm{d}s\right]. \end{aligned}$$

# Relationship Between Pontryagin and HJB Equation

Main object in Pontryagin maximum principle: Adjoint states $(p, q)$ and $(P, Q)$ given by BSDEs.
Main object in dynamic programming: Value function $V$ given by HJB equation.
Under smoothness assumptions, we have for an optimal trajectory $\bar{x}$

$$p_s = V_x(s, \bar{x}_s) \quad \text{and} \quad q_s = V_{xx}(s, \bar{x}_s)\sigma(\bar{x}_s, \bar{u}_s).$$

Open questions for infinite dimensional problems:

1. BSDE for second adjoint state;
2. Verification Theorem without smoothness assumptions;
3. Relationship between adjoint states and value function without smoothness assumptions;
4. Efficient approximation of BSDEs;
5. Efficient approximation of value function.

# Thank you for your attention!