Stochastic Optimal Control of the FitzHugh-Nagumo System

Wilhelm Stannat   Lukas Wessels

Technische Universität Berlin

wessels@math.tu-berlin.de

Workshop on Control of Self-Organizing Nonlinear Systems, Lutherstadt Wittenberg, August 21, 2019

Introduction

Consider the FitzHugh-Nagumo system

$$\begin{cases} \mathrm{d}v_t = [\Delta v_t + v_t(1-v_t)(v_t-\alpha) - w_t + I_t]\,\mathrm{d}t & \text{on } L^2(\Lambda) \\ \mathrm{d}w_t = \varepsilon(v_t - \gamma w_t + \delta)\mathrm{d}t & \text{on } L^2(\Lambda) \end{cases}$$

with Neumann boundary conditions, where $\Lambda \subset \mathbb{R}$ is a bounded interval, $\alpha \in (0,1)$, and ε, γ, $\delta \geq 0$. I_t: injected current.

Figure: Membrane potential v

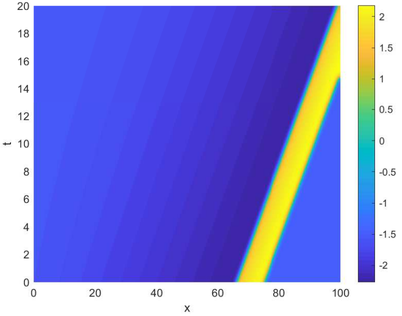
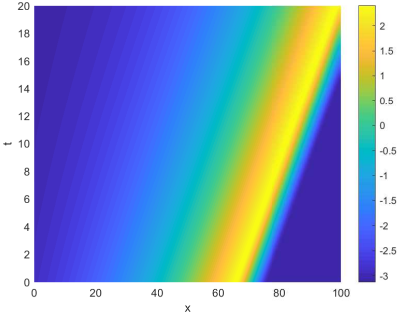


Figure: Recovery variable w



Motivation

Consider the FitzHugh-Nagumo system

$$\begin{cases} \mathrm{d}v_t = [\Delta v_t + v_t(1-v_t)(v_t-\alpha) - w_t + I_t]\,\mathrm{d}t + \sigma \mathrm{d}W_t & \text{on } L^2(\Lambda) \\ \mathrm{d}w_t = \varepsilon(v_t - \gamma w_t + \delta)\mathrm{d}t & \text{on } L^2(\Lambda) \end{cases}$$

with Neumann boundary conditions, where $\Lambda \subset \mathbb{R}$ is a bounded interval, $\alpha \in (0,1)$, and σ, ε, γ, $\delta \geq 0$. I_t: injected current.

Figure: Membrane potential v

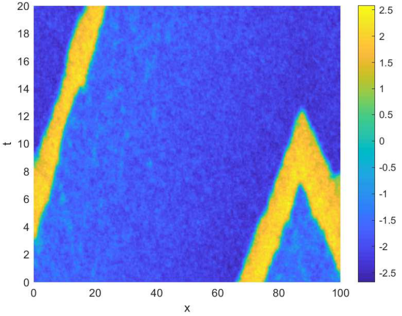
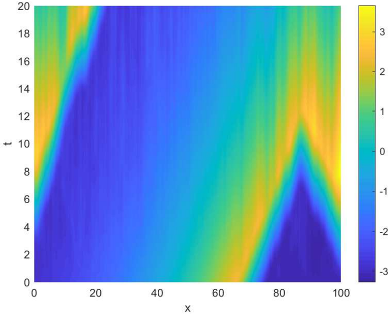


Figure: Recovery variable w



General Setting

With $V := H^1(\Lambda) \times L^2(\Lambda) \hookrightarrow H := L^2(\Lambda)^2$, $u = (v, w)$, this can be written as

$$\mathrm{d}u_t^g = [Au_t^g + F(t, u_t^g, g_t)]\,\mathrm{d}t + \sigma(t, u_t^g, g_t)\mathrm{d}W_t \qquad \text{on } H$$

Objective: Minimize

$$J(g) := \mathbb{E}\left[\int_0^T L(u_t^g, g_t)\mathrm{d}t + \Phi(u_T^g)\right]$$

over all

$$G_{\mathrm{ad}} := \left\{ g : [0, T] \times \Omega \to G : \sup_{t\in[0,T]} \|g_t\|_G \leq \kappa \text{ a.s. } , g\ (\mathcal{F}_t)_{t\geq 0} - \text{adapted} \right\},$$

Example: Quadratic costs:

$$L(u_t^g, g_t) = c\int_\Lambda \left(v_t^g(x) - v_\Lambda(t,x)\right)^2 \mathrm{d}x + \lambda \int_\Lambda g_t^2(x)\mathrm{d}x.$$

Assumptions

State equation:

$$\mathrm{d}u_t^g = [Au_t^g + F(t, u_t^g, g_t)]\,\mathrm{d}t + \sigma(t, u_t^g, g_t)\mathrm{d}W_t \qquad \text{on } H$$

$F : [0, T] \times V \times G \to V^*$ such that for all $u, \tilde{u}, v \in V$

$$_{V^*}\langle F(t, u, g) - F(t, \tilde{u}, g), u - \tilde{u}\rangle_V \leq C\|u - \tilde{u}\|_H^2$$

and

$$_{V^*}\langle \partial_u F(t, u, g)v, v\rangle_V \leq C(1 + \|v\|_H^2).$$

FOC in variational setting: Al-Hussein (Appl Math Optim, 2011)
FOC in mild setting: Fuhrmann et al. (Stoch PDE: Anal Comp, 2018)

⚠ FitzHugh-Nagumo system not included in those works.

Gâteaux Derivative of the Cost Functional

Proposition 1

The Gâteaux derivative of the cost functional in direction h is given by

$$\frac{\partial J(g)}{\partial h} = \mathbb{E}\left[\int_0^T \partial_u L(u_t^g, g_t) y_t^h + \partial_g L(u_t^g, g_t) h_t \mathrm{d}t + \partial_u \Phi(u_T^g) y_T^h\right],$$

where y^h is the solution of

$$\begin{cases} \mathrm{d}y_t^h = [Ay_t^h + \partial_u F(t, u_t^g, g_t) y_t^h + \partial_g F(t, u_t^g, g_t) h_t]\mathrm{d}t \\ \qquad\qquad + \left[\partial_u \sigma(t, u_t^g, g_t) y_t^h + \partial_g \sigma(t, u_t^g, g_t) h_t\right] \mathrm{d}W_t & \text{on } H \\ y_0^h = 0 & \text{in } H \end{cases}$$

Goal: Separate h in order to obtain a representation of the gradient, i.e.

$$\frac{\partial J(g)}{\partial h} = \langle \nabla J(g), h \rangle$$

for a suitable inner product $\langle \cdot, \cdot \rangle$. $\rightarrow$ Adjoint State

Adjoint State

Adjoint state in the deterministic case solves the backward PDE

$$\begin{cases} -\mathrm{d}p_t = [A^* p_t + \partial_u F(t, u_t^g, g_t)^* p_t + \partial_u L(u_t^g, g_t)]\mathrm{d}t & \text{on } H \\ p_T = \partial_u \Phi(u_T^g) & \text{in } H. \end{cases}$$

⚠ Adjoint state not adapted to the filtration of $(W_t)_{t \geq 0}$. In order to obtain an adapted solution, consider the backward SPDE

$$\begin{cases} -\mathrm{d}p_t = [A^* p_t + \partial_u F(t, u_t^g, g_t)^* p_t & \\ \qquad\qquad + \partial_u \sigma(t, u_t^g, g_t)^* P_t + \partial_u L(u_t^g, g_t)]\mathrm{d}t - P_t \mathrm{d}W_t & \text{on } H \\ p_T = \partial_u \Phi(u_T^g) & \text{in } H. \end{cases}$$

The solution is a pair of processes

$$(p, P) \in L^2([0, T] \times \Omega; V) \times L^2([0, T] \times \Omega; L_2(U, H)).$$

Backward Stochastic Differential Equations

Let $(W_t)_{t\geq 0}$ be a Brownian motion and $(\mathcal{F}_t)_{t\geq 0}$ its natural filtration. Consider the BSDE

$$\begin{cases} \mathrm{d}p_t = 0 \\ p_T = \xi, \end{cases}$$

where ξ is $\mathcal{F}_T$-measurable. ⚠ Adapted solution doesn't exist!
Natural candidate: $p_t = \mathbb{E}[\xi|\mathcal{F}_t]$. However, does not solve BSDE. Martingale representation theorem:

$$\mathbb{E}[\xi|\mathcal{F}_t] = \mathbb{E}[\xi] + \int_0^t P_s \mathrm{d}W_s.$$

Restate original problem:

$$\begin{cases} \mathrm{d}p_t = P_t \mathrm{d}W_t \\ p_T = \xi. \end{cases}$$

For $\xi = f(W_T)$:

$$\mathbb{E}\left[\xi|\mathcal{F}_t\right] = \mathbb{E}\left[f(W_T)|\mathcal{F}_t\right] = S_{T-t}f(W_t) = S_T f(W_0) + \int_0^t \nabla S_{T-s} f(W_s) \mathrm{d}W_s$$

Solving the Backward SPDEs

Consider finite dimensional approximations

$$\begin{cases} -\mathrm{d}p_t^n = [(\Pi_n A)^* p_t^n + (\Pi_n \partial_u F(t, u_t^g, g_t))^* p_t^n \\ \qquad\qquad + (\Pi_n \partial_u \sigma(t, u_t^g, g_t))^* P_t^n + \Pi_n \partial_u L(u_t^g, g_t)]\mathrm{d}t - P_t^n \mathrm{d}W_t & \text{on } H_n \\ p_T^n = \Pi_n \partial_u \Phi(u_T^g) & \text{in } H_n. \end{cases}$$

Derive a priori bounds:

Lemma 2

It holds

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\int_0^T \|p_t^n\|_V^2 \,\mathrm{d}t + \int_0^T \|P_t^n\|_{L_2(U,H)}^2 \,\mathrm{d}t\right] < \infty.$$

Extract weakly convergent subsequences and pass to the limit.

The Gradient of the Solution Map

$$\frac{\partial J(g)}{\partial h} = \mathbb{E}\left[\int_0^T \partial_u L(u_t^g, g_t) y_t^h + \partial_g L(u_t^g, g_t) h_t \mathrm{d}t + \partial_u \Phi(u_T^g) y_T^h\right]$$
$$= \mathbb{E}\left[\int_0^T \left(\partial_g F(t, u_t^g, g_t)^* p_t + \partial_g L(u_t^g, g_t) + \partial_g \sigma(t, u_t^g, g_t)^* P_t\right) h_t \mathrm{d}t\right].$$

Proposition 3

The gradient of the cost functional in $L^2([0,T] \times \Omega; G)$ is given by

$$\nabla J(g) = \partial_g F(t, u_t^g, g_t)^* p_t + \partial_g L(u_t^g, g_t) + \partial_g \sigma(t, u_t^g, g_t)^* P_t,$$

where (p, P) is the adjoint state.

How to use this result for applications?

$\rightarrow$ Restriction to deterministic controls: simpler random backward PDE.

Example

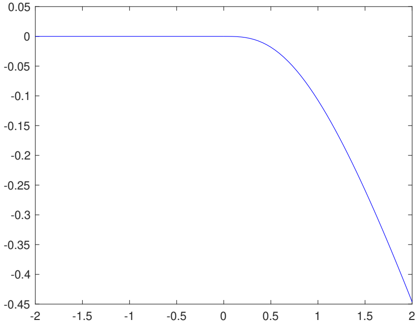
Consider the SDE

$$\begin{cases} \mathrm{d}u_t^g = [-V'(u_t^g) + g_t]\,\mathrm{d}t + \sigma \mathrm{d}B_t, & t \in [0,T] \\ u_0^g = 0 \end{cases}$$

where the potential V is given by

$$V(x) = \begin{cases} \frac{1}{2}(\arctan(x) - x), & \text{for } x \geq 0 \\ 0, & \text{for } x < 0, \end{cases}$$

i.e.,



Example

Consider the SDE

$$\begin{cases} \mathrm{d}u_t^g = [-V'(u_t^g) + g_t]\,\mathrm{d}t + \sigma \mathrm{d}B_t, \quad t \in [0, T] \\ u_0^g = 0. \end{cases}$$

We consider the cost functional

$$J(g) := \frac{1}{2}\mathbb{E}\left[\left(u_T^g\right)^2\right].$$

Leads to the adjoint equation

$$\begin{cases} -\partial_t p = -V''(u_t^g)p, \quad t \in [0, T] \\ p_T = u_T^g, \end{cases}$$

hence

$$p_t = u_T^g \exp\left(\int_t^T -V''(u_s^g)\mathrm{d}s\right).$$

Example

Consider the SDE

$$\begin{cases} \mathrm{d}u_t^g = [-V'(u_t^g) + g_t]\,\mathrm{d}t + \sigma \mathrm{d}B_t, \quad t \in [0, T] \\ u_0^g = 0. \end{cases}$$

The gradient of the cost functional is

$$\nabla J(g)(t) = \mathbb{E}[p_t] = \mathbb{E}\left[u_T^g \exp\left(\int_t^T -V''(u_s^g)\mathrm{d}s \right) \right].$$

Example

Consider the SDE

$$\begin{cases} \mathrm{d}u_t^g = [-V'(u_t^g) + g_t]\,\mathrm{d}t + \sigma \mathrm{d}B_t, \quad t \in [0, T] \\ u_0^g = 0. \end{cases}$$

Claim: The gradient for $g \equiv 0$ is not zero, hence $g \equiv 0$ is not an optimal control. Indeed,

$$\begin{aligned}
&\liminf_{t \to T} \{-\partial_t(\nabla J(g))(t)\} \\
=&\liminf_{t \to T} \mathbb{E}\left[-V''(u_t^g) u_T^g \exp\left(\int_t^T -V''(u_s^g)\mathrm{d}s \right) \right] \\
\geq&\mathbb{E}\left[\liminf_{t \to T} \left\{ -V''(u_t^g) u_T^g \exp\left(\int_t^T -V''(u_s^g)\mathrm{d}s \right) \right\} \right] \\
=&\mathbb{E}\left[-V''(u_T^g) u_T^g\right] \\
=&\mathbb{E}\left[\frac{(u_T^g)^2}{\left(1 + (u_T^g)^2\right)^2} 1_{\{u_T^g > 0\}} \right] > 0.
\end{aligned}$$

References

A. Al-Hussein
Necessary Conditions for Optimal Control of Stochastic Evolution Equations in Hilbert Spaces
Applied Mathematics & Optimization, 63:385–400, 2011

M. Fuhrmann, Y. Hu, G. Tessitore
Stochastic Maximum Principle for Optimal Control of Partial Differential Equations Driven by White Noise
Stochastics and Partial Differential Equations: Analysis and Computations, 6(2):255–285, 2018.

W. Stannat, L. Wessels
Deterministic Control of Stochastic Reaction-Diffusion Equations
Arxiv preprint arXiv:1905.09074, 2019. In review.

W. Stannat, L. Wessels
Stochastic Optimal Control of Reaction-Diffusion Systems
In preparation.