Optimal Control of the Stochastic Schlögl Equation

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Overview

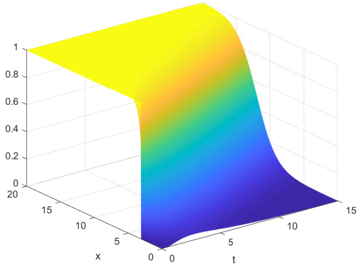
Introduction

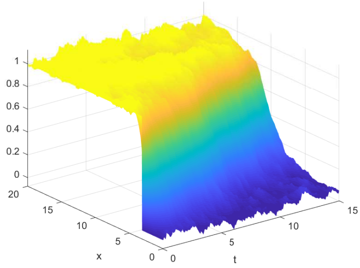
Motivation

Consider the Schlögl equation

$$\partial_t u = \Delta u + u(1-u)(u-a) + \xi \qquad \text{on } [0,T] \times \Lambda$$

with Neumann boundary conditions, where $\Lambda \subset \mathbb{R}$ is bounded and $a \in (0,1)$.





Choosing $\xi = \frac{\mathrm{d}W_t}{\mathrm{d}t}$, $(W_t)_{t\geq 0}$ $L^2(\Lambda)$-valued Wiener process, leads to

$$\mathrm{d}u_t = [\Delta u_t + u_t(1-u_t)(u_t - a)]\,\mathrm{d}t + \varepsilon \mathrm{d}W_t \qquad \text{on } L^2(\Lambda)$$

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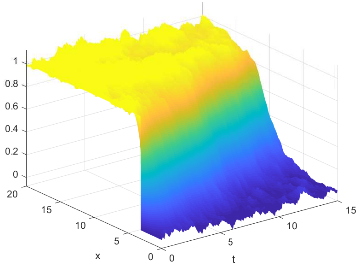

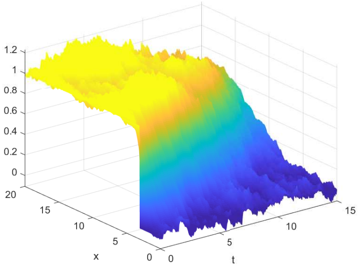

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Motivation

Introduce control $g : [0, T] \times \Lambda \to \mathbb{R}$:

$$\mathrm{d}u_t^g = [\Delta u_t^g + u_t^g(1-u_t^g)(u_t^g - a) + g(t)]\,\mathrm{d}t + \varepsilon \mathrm{d}W_t \qquad \text{on } L^2(\Lambda)$$

Objective: Minimize

$$J(g) := \mathbb{E}\left[\frac{c_{\overline{\Lambda}}}{2}\int_0^T \int_\Lambda \left(u_t^g(x) - u_{\overline{\Lambda}}(t,x)\right)^2 \mathrm{d}x\,\mathrm{d}t\right] + \mathbb{E}\left[\frac{c_T}{2}\int_\Lambda \left(u_T^g(x) - u^T(x)\right)^2 \mathrm{d}x\right] + \frac{\lambda}{2}\int_0^T \int_\Lambda g^2(t,x)\mathrm{d}x\mathrm{d}t$$

over all

$$g \in G_{\mathrm{ad}} = \left\{ g \in L^2\left([0,T]\times\Lambda\right) \mid \|g\|_{L^2([0,T]\times\Lambda)} \le \kappa \right\}.$$

General Setting

Consider the SPDE

$$\begin{aligned} \mathrm{d}u_t^g &= [\Delta u_t^g + f(u_t^g) + b(t)g(t)]\,\mathrm{d}t + \varepsilon \mathrm{d}W_t && \text{on } L^2(\Lambda) \\ u_0 &= v && \text{in } \Lambda \end{aligned}$$

with Neumann boundary conditions, where

- $v \in L^\infty(0, T)$
- $b \in L^\infty([0, T] \times \Lambda)$
- $f \in C^1$
- $\sup_{u \in \mathbb{R}} f'(u) < \infty$
- $|f'(u)| \leq C(1 + |u|^2) \qquad \forall u \in \mathbb{R}$

Deterministic case was considered by Engel/Tröltzsch et. al. in project B6.
Existence of optimal solution: Coayla-Teran et. al. (ArXiv Preprint, 2017).
First order conditions: Fuhrman et. al. (Stoch PDE: Anal Comp, 2018).

Well-Posedness of the Optimal Control Problem

Well-Posedness of the Control Problem

Proposition 1

There exist $C_1 = C_1(b, f, T)$ and $C_2 = C_2(f, T, Q)$ such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|u_t^g\|_{L^2(\Lambda)}^2 + \int_0^T \|\nabla u_t^g\|_{L^2(\Lambda)}^2 \,\mathrm{d}t\right] \leq C_1 \int_0^T \|g(t)\|_{L^2(\Lambda)}^2 \,\mathrm{d}t + C_2\varepsilon^2.$$

Sketch of the proof:

$$\|u_t^g\|_{L^2(\Lambda)}^2 = \|u_0^g\|_{L^2(\Lambda)}^2 + 2\int_0^t \langle \Delta u_s^g, u_s^g\rangle_{L^2(\Lambda)} \,\mathrm{d}s + 2\int_0^t \langle f\left(u_s^g\right), u_s^g\rangle_{L^2(\Lambda)} \,\mathrm{d}s$$

$$+ 2\int_0^t \langle b(s)g(s), u_s^g\rangle_{L^2(\Lambda)} \,\mathrm{d}s + 2\varepsilon \int_0^t \langle u_s^g, \mathrm{d}W_s\rangle_{L^2(\Lambda)} + \varepsilon^2 t \operatorname{tr} Q$$

Burkholder-Davis-Gundy-inequality:

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t \langle u_s^g, \mathrm{d}W_s\rangle_{L^2(\Lambda)}\right|\right] \leq C(Q)\,\mathbb{E}\left[\left(\int_0^T \|u_t^g\|_{L^2(\Lambda)}^2 \mathrm{d}t\right)^{\frac{1}{2}}\right].$$

Well-Posedness of the Control Problem

Remark: The solution map is Lipschitz continuous, i.e.

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|u_t^{g_1}-u_t^{g_2}\|_{L^2(\Lambda)}^2+\int_0^T\|\nabla(u_t^{g_1}-u_t^{g_2})\|_{L^2(\Lambda)}^2\,\mathrm{d}t\right]\lesssim\int_0^T\|g_1-g_2\|_{L^2(\Lambda)}^2\,\mathrm{d}s.$$

Theorem 2

There exists a $g^*\in G_{ad}$ such that $J(g^*)=\inf_{g\in G_{ad}}J(g)$.

Sketch of the proof:

- Minimizing sequence, weakly convergent subsequence $g_n\to g^*$ (as in the deterministic case).
- Apply compactness method for stochastic processes:
  - $\mathcal{L}(u^{g_n})$ are relatively compact.
  - Extract converging subsequence $\mathcal{L}(u^{g_n})\to\mathcal{L}(u^{g^*})$.
  - Identification of the limit $u^{g_n}\to u^{g^*}$.
- Exploit weak lower semicontinuity of J w.r.t. g.

The Gradient of the Cost Functional

The Gâteaux Derivative of the Solution Map

Proposition 3

Let $g \in L^2([0,T] \times \Lambda)$ be fixed. Then, for $h \in L^2([0,T] \times \Lambda)$, the Gâteaux derivative of the solution map in direction h is given by the variational solution of the equation

$$\begin{aligned} \partial_t y^h &= \Delta y^h + f'(u^g) y^h + bh && \text{on } [0,T] \times \Lambda \\ y^h(0,x) &= 0 && x \in \Lambda. \end{aligned}$$

Mild formulation of the Gâteaux derivative:

$$\frac{\partial u_t^g(x)}{\partial h} = y^h(t,x) = \int_0^t p_{s,t}^{u^g} \left(b(s,\cdot) h(s,\cdot) \right)(x) \mathrm{d}s,$$

where $(p_{s,t}^{u^g})_{0 \le s \le t \le T}$ denotes the random propagator of $\Delta + f'(u_t^g)$.

Computing the Gradient of the Cost Functional

The Gâteaux derivative of the cost functional:

$$
\begin{aligned}
\frac{\partial J(g)}{\partial h} =& \mathbb{E}\left[c_{\overline{\Lambda}} \int_0^T \int_\Lambda \int_0^t b(s,x)h(s,x)p_{s,t}^{u^g}\left(u_t^g(\cdot) - u_{\overline{\Lambda}}(t,\cdot)\right)\mathrm{d}s\mathrm{d}x\mathrm{d}t\right] \\
&+ \mathbb{E}\left[c_T \int_\Lambda \int_0^T b(s,x)h(s,x)p_{s,T}^{u^g}\left(u_T^g(\cdot) - u^T(\cdot)\right)\mathrm{d}s\mathrm{d}x\right] \\
&+ \lambda \int_0^T \int_\Lambda g(t,x)h(t,x)\mathrm{d}x\mathrm{d}t \\
=& \int_0^T \int_\Lambda \Bigg(\mathbb{E}\left[c_{\overline{\Lambda}}\, b(t,x) \int_t^T p_{t,s}^{u^g}\left(u_s^g(\cdot) - u_{\overline{\Lambda}}(s,\cdot)\right)(x)\mathrm{d}s\right] \\
&+ \mathbb{E}\left[c_T\, b(t,x)\, p_{t,T}^{u^g}\left(u_T^g(\cdot) - u^T(\cdot)\right)(x)\right] + \lambda g(t,x)\Bigg) h(t,x)\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

The Gradient of the Cost Functional

The Gradient of the cost functional J is given by

$$\nabla J(g)(t,x) = \mathbb{E}\left[c_{\overline{\Lambda}}\, b(t,x) \int_t^T p_{t,s}^{u^g}\left(u_s^g(\cdot) - u_{\overline{\Lambda}}(s,\cdot)\right)(x)\mathrm{d}s\right]$$

$$+\,\mathbb{E}\left[c_T\, b(t,x)\, p_{t,T}^{u^g}\left(u_T^g(\cdot) - u^T(\cdot)\right)(x)\right] + \lambda g(t,x).$$

We introduce the random backward PDE

$$\begin{aligned} -\partial_t q &= \Delta q + f'(u^g)q + c_{\overline{\Lambda}}\left(u^g - u_{\overline{\Lambda}}\right) && \text{on } [0,T]\times\Lambda \\ q(T,x) &= c_T\left(u_T^g(x) - u^T(x)\right) && x \in \Lambda, \end{aligned}$$

where the solution is given by

$$q(t,x) = c_{\overline{\Lambda}} \int_t^T p_{t,s}^{u^g}\left(u_s^g(\cdot) - u_{\overline{\Lambda}}(s,\cdot)\right)(x)\mathrm{d}s + c_T p_{t,T}^{u^g}\left(u_T^g(\cdot) - u^T(\cdot)\right)(x).$$

This gives us the representation

$$\nabla J(g)(t,x) = \mathbb{E}\left[b(t,x)q(t,x) + \lambda g(t,x)\right].$$

Gradient Descent Algorithm

Gradient Descent Algorithm

1. Solve state equation.
2. Solve random backward PDE (adjoint equation).
3. Repeat Step 1 and Step 2 to approximate

   $$\nabla J(g_n)(t,x) = \mathbb{E}\left[b(t,x)q_n(t,x) + \lambda g_n(t,x)\right]$$

   via Monte Carlo method.
4. Direction of descent: $d_n = -\nabla J(g_n)$.
5. Compute new control via $g_{n+1} = g_n + s d_n$.
6. Stop if $\|d_n\| < \delta$ or control hits the boundary, otherwise go to step 1.

Application to the Stochastic Schlögl Equation

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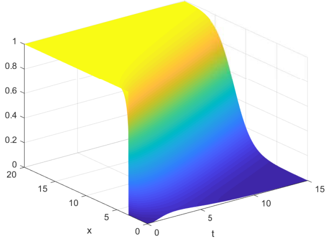
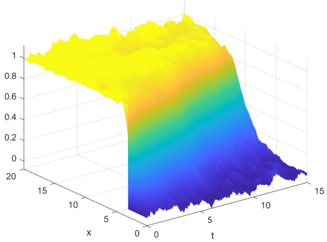
Recall the Schlögl equation:

$$\mathrm{d}u_t^g = [\Delta u_t^g + u_t^g(1-u_t^g)(u_t^g - a) + g(t)]\,\mathrm{d}t + \varepsilon \mathrm{d}W_t \qquad \text{on } L^2(\Lambda)$$





Parameters: $T = 15$, $\Lambda = [0, 20]$ and $a = 39/40$.

Objective: Control the wave.

Deterministic Case

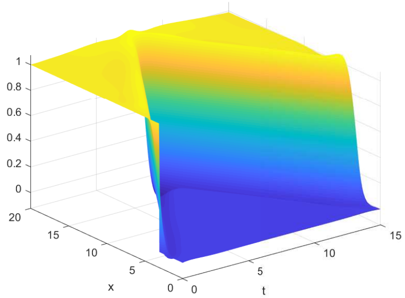


Figure: Controlled Solution

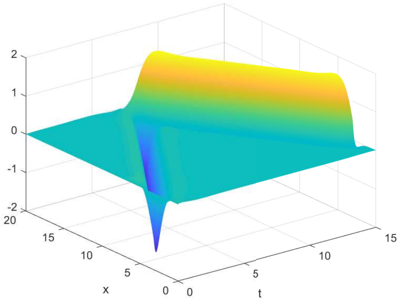
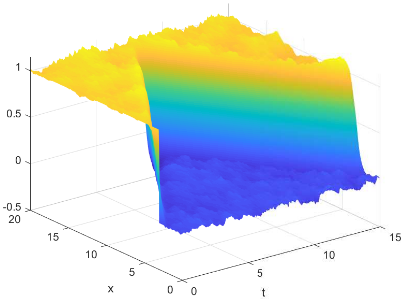


Figure: Optimal Control

Parameters: $T = 15$, $\Lambda = [0, 20]$ and $a = 39/40$.

Stochastic Case ($\varepsilon = 1$)



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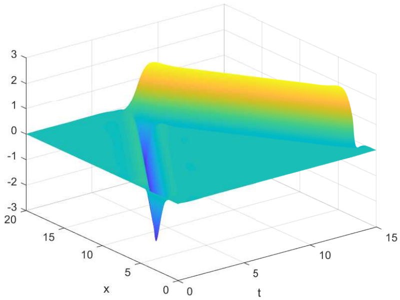

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Work in Progress.