Pontryagin's Maximum Principle for SPDEs and Its Relation to Dynamic Programming

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Motivation

Consider deterministic control problem: Minimize

$$J(u) := \int_0^T \int_\Lambda l(x_t^u(\lambda), u_t)\mathrm{d}\lambda\mathrm{d}t + \int_\Lambda h(x_T^u(\lambda))\mathrm{d}\lambda$$

subject to

$$\partial_t x_t^u = \Delta x_t^u + b(x_t^u, u_t).$$

Introduce random input disturbance $\xi_t(\lambda)$, i.e.

$$\partial_t x_t^u = \Delta x_t^u + b(x_t^u, u_t + \xi_t).$$

Taylor expansion yields

$$b(x_t^u, u_t + \xi_t) \approx b(x_t^u, u_t) + b_u(x_t^u, u_t)\xi_t.$$

Leads to controlled equation

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Taylor expansion yields

$$b(x_t^u, u_t + \xi_t) \approx b(x_t^u, u_t) + b_u(x_t^u, u_t)\xi_t.$$

Leads to controlled stochastic partial differential equation

$$\mathrm{d}x_t^u = [\Delta x_t^u + b(x_t^u, u_t)]\mathrm{d}t + b_u(x_t^u, u_t)\mathrm{d}W_t,$$

where $(W_t)_{t\geq 0}$ is an (infinite-dimensional) Brownian motion.

Setting

Minimize

$$J(u) := \mathbb{E}\left[\int_0^T \int_\Lambda l(x_t^u(\lambda), u_t)\mathrm{d}\lambda\mathrm{d}t + \int_\Lambda h(x_T^u(\lambda))\mathrm{d}\lambda\right]$$

subject to

$$\begin{cases} \mathrm{d}x_t^u = [\Delta x_t^u + b(x_t^u, u_t)]\mathrm{d}t + \sigma(x_t^u, u_t)\mathrm{d}W_t, \quad t \in [0, T] \\ x_0^u = x_0 \in L^2(\Lambda), \end{cases}$$

where

- l, h Nemytskii operators of (at most) quadratic growth;
- b, σ Nemytskii operators, Lipschitz;
- $(W_t)_{t\geq 0}$ cylindrical Wiener process;
- $\Lambda \subset \mathbb{R}^d$ bounded domain.

Goal: Characterize optimal controls.

Pontryagin 1956: Controlled ODEs

Minimize

$$J(u) := \int_0^T l(x_t^u, u_t)\mathrm{d}t + h(x_T^u)$$

subject to

$$\begin{cases} \partial_t x_t^u = b(x_t^u, u_t), & t \in [0, T] \\ x_0^u = x_0 \in \mathbb{R}^n. \end{cases}$$

Let $\bar{u}$ be optimal. Then

$$\inf_{u \in U} H(t, \bar{x}_t, u) = H(t, \bar{x}_t, \bar{u}_t),$$

for almost every $t \in [0, T]$, where Hamiltonian $H : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$

$$H(t, x, u) := l(x, u) + \langle p_t, b(x, u) \rangle$$

where p_t is the adjoint state.

Peng 1990: Controlled SDEs

Minimize

$$J(u) := \mathbb{E}\left[\int_0^T l(x_t^u, u_t)\mathrm{d}t + h(x_T^u)\right]$$

subject to

$$\begin{cases} \mathrm{d}x_t^u = b(x_t^u, u_t)\mathrm{d}t + \sigma(x_t^u, u_t)\mathrm{d}W_t, & t \in [0, T] \\ x_0^u = x_0 \in \mathbb{R}^n. \end{cases}$$

Let $\bar{u}$ be optimal. Then

$$\inf_{u \in U} \mathcal{G}(t, \bar{x}_t, u) = \mathcal{G}(t, \bar{x}_t, \bar{u}_t),$$

for almost all $(t, \omega) \in [0, T] \times \Omega$, where generalized Hamiltonian $\mathcal{G} : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$

$$\mathcal{G}(t, x, u) := H(t, x, u) + \frac{1}{2}\mathrm{tr}(\sigma(x, u)^* P_t \sigma(x, u)) + \mathrm{tr}\left(\sigma(x, u)^* \left[q_t - P_t \sigma(\bar{x}_t, \bar{u}_t)\right]\right)$$

where (p_t, q_t) first order adjoint state and $P_t \in \mathbb{R}^{n \times n}$ second order adjoint state.

Now: Controlled SPDEs

Minimize

$$J(u) := \mathbb{E}\left[\int_0^T \int_\Lambda l(x_t^u(\lambda), u_t)\mathrm{d}\lambda\mathrm{d}t + \int_\Lambda h(x_T^u(\lambda))\mathrm{d}\lambda\right]$$

subject to

$$\begin{cases} \mathrm{d}x_t^u = [\Delta x_t^u + b(x_t^u, u_t)]\mathrm{d}t + \sigma(x_t^u, u_t)\mathrm{d}W_t, \quad t \in [0, T] \\ x_0^u = x_0 \in L^2(\Lambda). \end{cases}$$

- Since Peng's result: Many generalizations to infinite-dimensional stochastic case by Du, Fuhrman, Frankowska, Guatteri, Hu, Li, Lü, Meng, Tang, Tessitore, Zhang, ...
- Major drawback in previous works: Restrictive assumptions on coefficients l and h, in particular excluding quadratic costs.
- Main obstacle: Characterization of second order adjoinst state $P_t \in L(H)$.

Taylor Expansions of the Cost Functional

First Order Expansion of Cost Functional

Let $\bar{u}$ be optimal. Fix $\tau \in [0, T]$, $\varepsilon > 0$, $v \in U$, and set

$$u_t^\varepsilon := \begin{cases} v, & \tau \le t \le \tau + \varepsilon, \\ \bar{u}_t & \text{otherwise.} \end{cases}$$

Then

$$0 \le J(u^\varepsilon) - J(\bar{u}) = \mathbb{E}\left[\int_0^T \int_\Lambda l(x_t^\varepsilon, u_t^\varepsilon) - l(\bar{x}_t, \bar{u}_t) \mathrm{d}\lambda \mathrm{d}t + \int_\Lambda h(x_T^\varepsilon) - h(\bar{x}_T) \mathrm{d}\lambda\right].$$

Taylor expansions: For terminal costs we have

$$\mathbb{E}\left[\int_\Lambda h(x_T^\varepsilon) - h(\bar{x}_T) \mathrm{d}\lambda\right] \approx \mathbb{E}\left[\int_\Lambda h_x(\bar{x}_T) y_T^\varepsilon \mathrm{d}\lambda\right],$$

where y^ε satisfies linearized state equation

$$\begin{cases} \mathrm{d}y_t^\varepsilon = \left[\Delta y_t^\varepsilon + b_x(\bar{x}_t, \bar{u}_t) y_t^\varepsilon + b(\bar{x}_t, u_t^\varepsilon) - b(\bar{x}_t, \bar{u}_t)\right] \mathrm{d}t \\ \qquad\qquad + \left[\sigma_x(\bar{x}_t, \bar{u}_t) y_t^\varepsilon + \sigma(\bar{x}_t, u_t^\varepsilon) - \sigma(\bar{x}_t, \bar{u}_t)\right] \mathrm{d}W_t \\ y_0^\varepsilon = 0. \end{cases}$$

Asymptotic of First Order Taylor Expansions

Applying Itô's formula to $\|y_t^\varepsilon\|^2_{L^2(\Lambda)}$ yields

$$\begin{aligned}\|y_t^\varepsilon\|^2_{L^2(\Lambda)} =& 2\int_0^t \langle \Delta y_s^\varepsilon + b_x(\bar{x}_s, \bar{u}_s) y_s^\varepsilon + b(\bar{x}_s, u_s^\varepsilon) - b(\bar{x}_s, \bar{u}_s), y_s^\varepsilon \rangle \mathrm{d}s \\ &+ 2\int_0^t \langle y_s^\varepsilon, (\sigma_x(\bar{x}_s, \bar{u}_s) y_s^\varepsilon + \sigma(\bar{x}_s, u_s^\varepsilon) - \sigma(\bar{x}_s, \bar{u}_s)) \mathrm{d}W_s \rangle \\ &+ \int_0^t \|\sigma_x(\bar{x}_s, \bar{u}_s) y_s^\varepsilon + \sigma(\bar{x}_s, u_s^\varepsilon) - \sigma(\bar{x}_s, \bar{u}_s)\|^2_{L_2(\Xi, L^2(\Lambda))} \mathrm{d}s.\end{aligned}$$

Itô correction term

$$\begin{aligned}&\int_0^t \|\sigma_x(\bar{x}_s, \bar{u}_s) y_s^\varepsilon + \sigma(\bar{x}_s, u_s^\varepsilon) - \sigma(\bar{x}_s, \bar{u}_s)\|^2_{L_2(\Xi, L^2(\Lambda))} \mathrm{d}s \\ \leq& 2\int_0^t \|\sigma_x(\bar{x}_s, \bar{u}_s) y_s^\varepsilon\|^2_{L_2(\Xi, L^2(\Lambda))} \mathrm{d}s + 2\int_\tau^{\tau+\varepsilon} \|\sigma(\bar{x}_s, v) - \sigma(\bar{x}_s, \bar{u}_s)\|^2_{L_2(\Xi, L^2(\Lambda))} \mathrm{d}s.\end{aligned}$$

The last term is of order $\mathcal{O}(\varepsilon)$; we need $o(\varepsilon)$. $\rightsquigarrow$ need second order Taylor expansions!

Variational Equations

First order:

$$\begin{cases} \mathrm{d}y_t^\varepsilon = [\Delta y_t^\varepsilon + b_x(\bar{x}_t, \bar{u}_t)y_t^\varepsilon + b(\bar{x}_t, u_t^\varepsilon) - b(\bar{x}_t, \bar{u}_t)]\,\mathrm{d}t \\ \qquad\qquad + [\sigma_x(\bar{x}_t, \bar{u}_t)y_t^\varepsilon + \sigma(\bar{x}_t, u_t^\varepsilon) - \sigma(\bar{x}_t, \bar{u}_t)]\,\mathrm{d}W_t \\ y_0^\varepsilon = 0. \end{cases}$$

Second order:

$$\begin{cases} \mathrm{d}z_t^\varepsilon = \left[\Delta z_t^\varepsilon + b_x(\bar{x}_t, \bar{u}_t)z_t^\varepsilon + \frac{1}{2}b_{xx}(\bar{x}_t, \bar{u}_t)y_t^\varepsilon y_t^\varepsilon + (b_x(\bar{x}_t, u_t^\varepsilon) - b_x(\bar{x}_t, \bar{u}_t))y_t^\varepsilon\right]\mathrm{d}t \\ \qquad\qquad + \left[\sigma_x(\bar{x}_t, \bar{u}_t)z_t^\varepsilon + \frac{1}{2}\sigma_{xx}(\bar{x}_t, \bar{u}_t)y_t^\varepsilon y_t^\varepsilon + (\sigma_x(\bar{x}_t, u_t^\varepsilon) - \sigma_x(\bar{x}_t, \bar{u}_t))y_t^\varepsilon\right]\mathrm{d}W_t \\ z_0^\varepsilon = 0. \end{cases}$$

Lemma

It holds that

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|y_t^\varepsilon\|_{L^2(\Lambda)}^2\right] \le C\varepsilon \qquad \textit{and} \qquad \sup_{t\in[0,T]} \mathbb{E}\left[\|z_t^\varepsilon\|_{L^2(\Lambda)}^2\right] \le C\varepsilon^2.$$

Second Order Expansion of Cost Functional

From

$$0 \leq J(u^\varepsilon) - J(\bar{u}) = \mathbb{E}\left[\int_0^T \int_\Lambda l(x_t^\varepsilon, u_t^\varepsilon) - l(\bar{x}_t, \bar{u}_t)\mathrm{d}\lambda\mathrm{d}t + \int_\Lambda h(x_T^\varepsilon) - h(\bar{x}_T)\mathrm{d}\lambda\right]$$

we derive:

Lemma

It holds

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \int_\Lambda l_x(\bar{x}_t(\lambda), \bar{u}_t)(y_t^\varepsilon(\lambda) + z_t^\varepsilon(\lambda)) + \frac{1}{2} l_{xx}(\bar{x}_t(\lambda), \bar{u}_t) y_t^\varepsilon(\lambda) y_t^\varepsilon(\lambda)\mathrm{d}\lambda\mathrm{d}t\right] \\
&+\mathbb{E}\left[\int_\Lambda h_x(\bar{x}_T(\lambda))(y_T^\varepsilon(\lambda) + z_T^\varepsilon(\lambda)) + \frac{1}{2} h_{xx}(\bar{x}_T(\lambda)) y_T^\varepsilon(\lambda) y_T^\varepsilon(\lambda)\mathrm{d}\lambda\right] \\
&+\mathbb{E}\left[\int_0^T \int_\Lambda l(\bar{x}_t(\lambda), u_t^\varepsilon) - l(\bar{x}_t(\lambda), \bar{u}_t)\mathrm{d}\lambda\mathrm{d}t\right] \geq o(\varepsilon).
\end{aligned}$$

Next step: Separate dependence on ε.

Adjoint States

First Order Adjoint State

Consider SPDE

$$\begin{cases} \mathrm{d}y_t = [\Delta y_t + b_x(\bar{x}_t, \bar{u}_t)y_t + \varphi_t]\,\mathrm{d}t + [\sigma_x(\bar{x}_t, \bar{u}_t)y_t + \psi_t]\,\mathrm{d}W_t \\ y_0 = 0, \end{cases}$$

where $(\varphi, \psi) \in L^2([0,T] \times \Omega; L^2(\Lambda)) \times L^2([0,T] \times \Omega; L_2(\Xi, L^2(\Lambda)))$.
Construct linear functional

$$\mathcal{T}(\varphi, \psi) := \mathbb{E}\left[\int_0^T \int_\Lambda l_x(\bar{x}_t(\lambda), \bar{u}_t)y_t(\lambda)\mathrm{d}\lambda\mathrm{d}t + \int_\Lambda h_x(\bar{x}_T(\lambda))y_T(\lambda)\mathrm{d}\lambda\right].$$

By Riesz's representation theorem, there is a unique pair

$$(p, q) \in L^2([0,T] \times \Omega; L^2(\Lambda)) \times L^2([0,T] \times \Omega; L_2(\Xi, L^2(\Lambda)))$$

such that

$$\mathcal{T}(\varphi, \psi) = \mathbb{E}\left[\int_0^T \langle \varphi_t, p_t\rangle_{L^2(\Lambda)} + \langle \psi_t, q_t\rangle_{L_2(\Xi, L^2(\Lambda))}\mathrm{d}t\right]$$

for all $(\varphi, \psi) \in L^2([0,T] \times \Omega; L^2(\Lambda)) \times L^2([0,T] \times \Omega; L_2(\Xi, L^2(\Lambda)))$.

First Order Adjoint Equation

Adjoint state property:

$$\mathbb{E}\left[\int_0^T \langle y_t, l_x(\bar{x}_t, \bar{u}_t)\rangle \mathrm{d}t + \langle h_x(\bar{x}_T), y_T\rangle\right] = \mathbb{E}\left[\int_0^T \langle p_t, \varphi_t\rangle + \langle q_t, \psi_t\rangle \mathrm{d}t\right].$$

Applying Itô's product rule yields

$$\begin{aligned}\mathrm{d}\langle p_t, y_t\rangle_{L^2(\Lambda)} =& \langle p_t, \mathrm{d}y_t\rangle_{L^2(\Lambda)} + \langle y_t, \mathrm{d}p_t\rangle_{L^2(\Lambda)} + \mathrm{d}\langle p, y\rangle_t \\ \overset{!}{=}& \left[\langle p_t, \varphi_t\rangle_{L^2(\Lambda)} + \langle q_t, \psi_t\rangle_{L_2(\Xi, L^2(\Lambda))} - \langle y_t, l_x(\bar{x}_t, \bar{u}_t)\rangle_{L^2(\Lambda)}\right] \mathrm{d}t + dM_t.\end{aligned}$$

for some martingale $(M_t)_{t\geq 0}$. Thus,

$$\begin{cases}\mathrm{d}p_t = -\left[\Delta p_t + b_x(\bar{x}_t, \bar{u}_t)p_t + l_x(\bar{x}_t, \bar{u}_t) + \langle \sigma_x(\bar{x}_t, \bar{u}_t), q_t\rangle_{L_2(\Xi, \mathbb{R})}\right] \mathrm{d}t + q_t \mathrm{d}W_t \\ p_T = h_x(\bar{x}_T).\end{cases}$$

Unique variational solution (p, q), where

$$p \in L^2([0,T]\times\Omega; H_0^1(\Lambda)) \cap L^2(\Omega; C([0,T]; L^2(\Lambda)))$$

and

$$q \in L^2([0,T]\times\Omega; L_2(\Xi, L^2(\Lambda))).$$

Backward Stochastic Differential Equations

Let $(W_t)_{t\geq 0}$ be a Brownian motion and $(\mathcal{F}_t)_{t\geq 0}$ its natural filtration. Consider the BSDE

$$\begin{cases} \mathrm{d}p_t = 0 \\ p_T = \xi, \end{cases}$$

where ξ is $\mathcal{F}_T$-measurable. ⚠ Adapted solution doesn't exist!
Natural candidate: $p_t = \mathbb{E}[\xi|\mathcal{F}_t]$. However, does not solve BSDE. Martingale representation theorem:

$$\mathbb{E}[\xi|\mathcal{F}_t] = \mathbb{E}[\xi] + \int_0^t q_s \mathrm{d}W_s.$$

Restate original problem:

$$\begin{cases} \mathrm{d}p_t = q_t \mathrm{d}W_t \\ p_T = \xi. \end{cases}$$

For $\xi = f(W_T)$:

$$\mathbb{E}\left[\xi|\mathcal{F}_t\right] = \mathbb{E}\left[f(W_T)|\mathcal{F}_t\right] = S_{T-t}f(W_t) = S_T f(W_0) + \int_0^t \nabla S_{T-s}f(W_s)\mathrm{d}W_s$$

Quadratic Terms

Follow same route for quadratic terms

$$\mathbb{E}\left[\int_0^T \int_\Lambda l_{xx}(\bar{x}_t(\lambda), \bar{u}_t) y_t^\varepsilon(\lambda) y_t^\varepsilon(\lambda) \mathrm{d}\lambda \mathrm{d}t + \int_\Lambda h_{xx}(\bar{x}_T(\lambda)) y_T^\varepsilon(\lambda) y_T^\varepsilon(\lambda) \mathrm{d}\lambda\right].$$

Peng's idea in finite dimensions: Linearize using tensor product $y_t^\varepsilon \otimes y_t^\varepsilon$ and derive equation on

$$\mathbb{R}^n \otimes \mathbb{R}^n \cong \mathbb{R}^{n \times n}.$$

Infinite dimensional analogue:

$$H \otimes H \cong L_2(H).$$

Problem: In order to perform duality analysis, we need to solve equation in $L_1(H)$.

Explicit Tensor Product

Use explicit representation

$$L^2(\Lambda) \otimes L^2(\Lambda) \cong L^2(\Lambda^2)$$
$$y \otimes z \leftrightsquigarrow ((\lambda, \mu) \mapsto y(\lambda)z(\mu)).$$

Rewrite quadratic terms as

$$\mathbb{E}\left[\int_0^T \int_\Lambda l_{xx}(\bar{x}_t(\lambda), \bar{u}_t) y_t^\varepsilon(\lambda) y_t^\varepsilon(\lambda) \mathrm{d}\lambda \mathrm{d}t\right]$$
$$= \mathbb{E}\left[\int_0^T \int_\Lambda l_{xx}(\bar{x}_t(\lambda), \bar{u}_t) \delta(y_t^\varepsilon \otimes y_t^\varepsilon)(\lambda) \mathrm{d}\lambda \mathrm{d}t\right],$$

where $\delta : H_0^1(\Lambda^2) \to L^2(\Lambda)$ is defined by $\delta(w)(\lambda) := w(\lambda, \lambda)$. This expression is linear in $y_t^\varepsilon \otimes y_t^\varepsilon$.

Second Order Adjoint Equation

Theorem (Stannat, W. (2021))

The equation

$$\begin{cases} \mathrm{d}P_t(\lambda,\mu) = -[\Delta P_t(\lambda,\mu) + (b_x(\bar{x}_t(\lambda),\bar{u}_t) + b_x(\bar{x}_t(\mu),\bar{u}_t))P_t(\lambda,\mu) \\ \qquad +\delta^*(l_{xx}(\bar{x}_t(\lambda),\bar{u}_t)) + \delta^*(b_{xx}(\bar{x}_t(\lambda),\bar{u}_t)p_t(\lambda)) \\ \qquad +\langle \sigma_x(\bar{x}_t(\lambda),\bar{u}_t), \sigma_x(\bar{x}_t(\mu),\bar{u}_t)\rangle_{L_2(\Xi,\mathbb{R})} P_t(\lambda,\mu) \\ \qquad +\langle \sigma_x(\bar{x}_t(\lambda),\bar{u}_t) + \sigma_x(\bar{x}_t(\mu),\bar{u}_t), Q_t(\lambda,\mu)\rangle_{L_2(\Xi,\mathbb{R})} \\ \qquad +\delta^*(\langle \sigma_{xx}(\bar{x}_t(\lambda),\bar{u}_t), q_t\rangle_{L_2(\Xi,\mathbb{R})})]\mathrm{d}t + Q_t(\lambda,\mu)\mathrm{d}W_t \\ P_T(\lambda,\mu) = \delta^*(h_{xx}(\bar{x}_T(\lambda))) \end{cases}$$

has a unique adapted solution (P, Q), *where*

$$P \in L^2([0,T]\times\Omega; L^2(\Lambda^2)) \cap L^2(\Omega; C([0,T]; H^{-1}(\Lambda^2))),$$

and

$$Q \in L^2([0,T]\times\Omega; L_2(\Xi; H^{-1}(\Lambda^2))).$$

Pontryagin's Maximum Principle for SPDEs

Theorem (Stannat, W. (2021))

Let $(\bar{x}, \bar{u})$ be an optimal pair. Then there exist adapted processes

$$(p, q) \in L^2([0, T] \times \Omega; H_0^1(\Lambda)) \times L^2([0, T] \times \Omega; L_2(\Xi, L^2(\Lambda)))$$

satisfying the first order adjoint equation and adapted processes

$$(P, Q) \in L^2([0, T] \times \Omega; L^2(\Lambda^2)) \times L^2([0, T] \times \Omega; L_2(\Xi, H^{-1}(\Lambda^2)))$$

satisfying the second order adjoint equation such that

$$\inf_{u \in U} \mathcal{G}(t, \bar{x}_t, u) = \mathcal{G}(t, \bar{x}_t, \bar{u}_t),$$

for almost all $(t, \omega) \in [0, T] \times \Omega$, where $\mathcal{G} : [0, T] \times L^2(\Lambda) \times U \to \mathbb{R}$

$$\begin{aligned}\mathcal{G}(t, x, u) := \int_\Lambda l(x(\lambda), u) \mathrm{d}\lambda + \langle p_t, b(x, u) \rangle_{L^2(\Lambda)} + \frac{1}{2} tr(\sigma(x, u)^* P_t \sigma(x, u)) \\ + tr\left(\sigma(x, u)^* \left[q_t - P_t \sigma(\bar{x}_t, \bar{u}_t)\right]\right).\end{aligned}$$

Connection with Dynamic Programming

Dynamic Programming Approach

Let $s \in [0, T]$. Minimize

$$J(s, x; u) := \mathbb{E}\left[\int_s^T \int_\Lambda l(x_t^u(\lambda), u_t)\mathrm{d}\lambda\mathrm{d}t + \int_\Lambda h(x_T^u(\lambda))\mathrm{d}\lambda\right]$$

over $u \in \mathcal{U}_s$ subject to

$$\begin{cases} \mathrm{d}x_t^u = [\Delta x_t^u + b(x_t^u, u_t)]\mathrm{d}t + \sigma(x_t^u, u_t)\mathrm{d}W_t, \quad t \in [s, T] \\ x_s^u = x \in L^2(\Lambda). \end{cases}$$

Introduce value function $V : [0, T] \times L^2(\Lambda) \to \mathbb{R}$,

$$V(s, x) := \inf_{u \in \mathcal{U}_s} J(s, x; u).$$

Satisfies dynamic programming principle

$$V(s, x) = \inf_{u \in \mathcal{U}_s} \mathbb{E}\left[\int_s^\tau l(x_t^u, u_t)\mathrm{d}t + V(\tau, x_\tau^u)\right], \qquad \forall \tau \in [s, T].$$

Can be used to derive optimality conditions.

Relationship under Smoothness Assumptions

Theorem (Bismut (1978))

If $V \in C^{1,3}([0,T] \times \mathbb{R}^d)$ *and* V_{sx} *is continuous, then for almost all* $t \in [s,T]$

1. $\begin{cases} V_x(t,\bar{x}_t) = p_t \\ V_{xx}(t,\bar{x}_t)\sigma(\bar{x}_t,\bar{u}_t) = q_t \end{cases}$
2. $-V_s(t,\bar{x}_t) = \inf_{u\in U} \mathcal{H}(\bar{x}_t, u, V_x(t,\bar{x}_t), V_{xx}(t,\bar{x}_t))$
3. $\inf_{u\in U} \mathcal{H}(\bar{x}_t, u, V_x(t,\bar{x}_t), V_{xx}(t,\bar{x}_t)) = \mathcal{H}(\bar{x}_t, \bar{u}_t, V_x(t,\bar{x}_t), V_{xx}(t,\bar{x}_t))$

where

$$\mathcal{H}(x,u,p,P) := \int_\Lambda l(x(\lambda),u)\mathrm{d}\lambda + \langle p, b(x,u)\rangle_{L^2(\Lambda)} + \frac{1}{2} tr(\sigma(x,u)^* P \sigma(x,u)).$$

Problem: In general, V is not differentiable!

Parabolic Viscosity Superdifferential

If $V \in C^{1,2}([0,T] \times L^2(\Lambda))$, it holds

$$\lim_{\tau \downarrow t, z \to x} \frac{1}{|\tau - t| + \|z - x\|^2} \Big[V(\tau, z) - V(t,x) - \partial_t V(t,x)(\tau - t) - \langle DV(t,x), z - x \rangle_{L^2(\Lambda)} - \frac{1}{2} \langle z - x, D^2 V(t,x)(z - x) \rangle_{L^2(\Lambda)} \Big] = 0.$$

Weaker notion of differentiability: We say $(G, p, P) \in D^{1,2,+}_{t+,x} V(t,x)$ if

$$\limsup_{\tau \downarrow t, z \to x} \frac{1}{|\tau - t| + \|z - x\|^2} \Big[V(\tau, z) - V(t,x) - G(\tau - t) - \langle p, z - x \rangle_{L^2(\Lambda)} - \frac{1}{2} \langle z - x, P(z - x) \rangle_{L^2(\Lambda)} \Big] \leq 0.$$

Theorem (Stannat, W. (2021+))

For almost every $t \in [0,T]$, it holds that

$$[-\langle \Delta \bar{x}_t, p_t \rangle_{H^{-1}(\Lambda) \times H^1_0(\Lambda)} - \mathcal{G}(t, \bar{x}_t, \bar{u}_t), \infty) \times \{p_t\} \times [P_t, \infty) \subset D^{1,2,+}_{t+,x} V(t, \bar{x}_t)$$

$\mathbb{P}$-almost surely.

Sketch of the Proof

Let $x^{\tau,z}$ be the solution to

$$\begin{cases} \mathrm{d}x_r^{\tau,z} = [\Delta x_r^{\tau,z} + b(x_r^{\tau,z}, \bar{u}_r)]\,\mathrm{d}r + \sigma(x_r^{\tau,z}, \bar{u}_r)\mathrm{d}W_r, \quad r \in [\tau, T] \\ x_\tau^{\tau,z} = z \in L^2(\Lambda). \end{cases}$$

Consider the difference

$$\begin{aligned} & V(\tau, z) - V(t, \bar{x}_t) \\ \leq & J(\tau, z; \bar{u}) - V(t, \bar{x}_t) \\ = & \mathbb{E}\left[-\int_t^\tau \int_\Lambda l(\bar{x}_r(\lambda), \bar{u}_r)\mathrm{d}\lambda\mathrm{d}r + \int_\tau^T \int_\Lambda l(x_r^{\tau,z}(\lambda), \bar{u}_r) - l(\bar{x}_r(\lambda), \bar{u}_r)\mathrm{d}\lambda\mathrm{d}r \middle| \mathcal{F}_t^s \right] \\ & + \mathbb{E}\left[\int_\Lambda h(x_T^{\tau,z}(\lambda)) - h(\bar{x}_T(\lambda))\mathrm{d}\lambda \middle| \mathcal{F}_t^s \right]. \end{aligned}$$

Next steps:

1. Employ Taylor expansions;
2. Plug in duality relations;
3. Rearrange terms, divide by $|\tau - t| + \|z - \bar{x}_t\|^2$, and take $\limsup_{\tau \downarrow t, z \to \bar{x}_t}$. □

Hamilton-Jacobi-Bellman Equation

Viscosity Solutions

Value function formally satisfies HJB equation

$$\begin{cases} V_s + \langle \Delta x, DV \rangle_{L^2(\Lambda)} + \inf_{u \in U} \mathcal{H}(x, u, DV, D^2 V) = 0, & (s, x) \in [0, T] \times L^2(\Lambda) \\ V(T, x) = \int_\Lambda h(x(\lambda)) \mathrm{d}\lambda, & x \in L^2(\Lambda) \end{cases}$$

where

$$\mathcal{H}(x, u, p, P) := \int_\Lambda l(x(\lambda), u) \mathrm{d}\lambda + \langle p, b(x, u) \rangle_{L^2(\Lambda)} + \frac{1}{2} \mathrm{tr} \left(\sigma(x, u)^* P \sigma(x, u) \right).$$

Definition (Viscosity Solution, Bounded Case)

V is viscosity subsolution, if

- $V(T, x) \leq \int_\Lambda h(x(\lambda)) \mathrm{d}\lambda, \quad x \in L^2(\Lambda);$
- *for every* $(G, p, P) \in D^{1,2,+}_{t,x} V(t, x)$

$$G + \langle Ax, p \rangle + \inf_{u \in U} \mathcal{H}(x, u, p, P) \leq 0.$$

Viscosity Solutions II

It holds:

$$\boxed{(G, p, P) \in D^{1,2,+}_{t,x} v(t,x)}$$

$$\Longleftrightarrow$$

$\exists \phi \in C^{1,2}((s,T) \times L^2(\Lambda))$ such that:

1. $v - \phi$ attains maximum at (t,x),
2. $(\phi(t,x), \partial_t \phi(t,x), D\phi(t,x), D^2\phi(t,x)) = (v(t,x), G, p, P)$.

Equivalent definition of viscosity solution in the bounded case (!):

Definition (Viscosity Solution, Bounded Case)

V is a viscosity subsolution, if

- $V(T,x) \leq \int_\Lambda h(x(\lambda))\mathrm{d}\lambda, \quad x \in L^2(\Lambda);$
- $\forall \phi \in C^{1,2}((s,T) \times L^2(\Lambda))$ *such that* $V - \phi$ *attains maximum at* (t,x), *it holds*

$$\phi_s(t,x) + \langle Ax, D\phi(t,x)\rangle + \inf_{u \in U} \mathcal{H}(x, u, D\phi(t,x), D^2\phi(t,x)) \leq 0.$$

Generalized Hamiltonian vs. Hamiltonian

Corollary (Stannat, W. (2021+))

It holds for almost all $t \in [s, T]$

$$\mathcal{G}(t, \bar{x}_t, \bar{u}_t) \leq \mathcal{H}(t, \bar{x}_t, \bar{u}_t, p_t, P_t),$$

$\mathbb{P}$*-almost surely, i.e.*

$$tr(\sigma(\bar{x}_t, \bar{u}_t)(q_t - P_t \sigma(\bar{x}_t, \bar{u}_t))) \leq 0.$$

Proof in the Bounded Case

Since

$$\left(-\langle A\bar{x}_t, p_t\rangle_{H^{-1}(\Lambda)\times H_0^1(\Lambda)} - \mathcal{G}(t,\bar{x}_t,\bar{u}_t), p_t, P_t\right) \in D^{1,2,+}_{t+,x} V(t,\bar{x}_t),$$

there exists a test function ϕ satisfying

$$(\phi(t,\bar{x}_t), \phi_t(t,\bar{x}_t), D\phi(t,\bar{x}_t), D^2\phi(t,\bar{x}_t)) = (V(t,\bar{x}_t), -\langle A\bar{x}_t, p_t\rangle - \mathcal{G}(t,\bar{x}_t,\bar{u}_t), p_t, P_t).$$

Since V is a viscosity solution of the HJB equation, we have

$$\begin{aligned}
0 &\leq \phi_t(t,\bar{x}_t) + \langle A\bar{x}_t, D\phi(t,\bar{x}_t)\rangle + \inf_{u\in U} \mathcal{H}(t,\bar{x}_t,u,D\phi(t,\bar{x}_t),D^2\phi(t,\bar{x}_t))\\
&= -\mathcal{G}(t,\bar{x}_t,\bar{u}_t) - \langle A\bar{x}_t, p_t\rangle + \langle A\bar{x}_t, p_t\rangle + \inf_{u\in U} \mathcal{H}(t,\bar{x}_t,u,p_t,P_t)\\
&= -\mathcal{H}(t,\bar{x}_t,\bar{u}_t,p_t,P_t) + \inf_{u\in U} \mathcal{H}(t,\bar{x}_t,u,p_t,P_t) - \operatorname{tr}\left(\sigma(\bar{x}_t,\bar{u}_t)^*(q_t - P_t\sigma(\bar{x}_t,\bar{u}_t))\right)\\
&\leq -\operatorname{tr}\left(\sigma(\bar{x}_t,\bar{u}_t)^*(q_t - P_t\sigma(\bar{x}_t,\bar{u}_t))\right).
\end{aligned}$$

□

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