Optimal Control of Stochastic Reaction-Diffusion Equations

Lukas Wessels

joint work with Wilhelm Stannat

Technische Universität Berlin

www.lukaswessels.org

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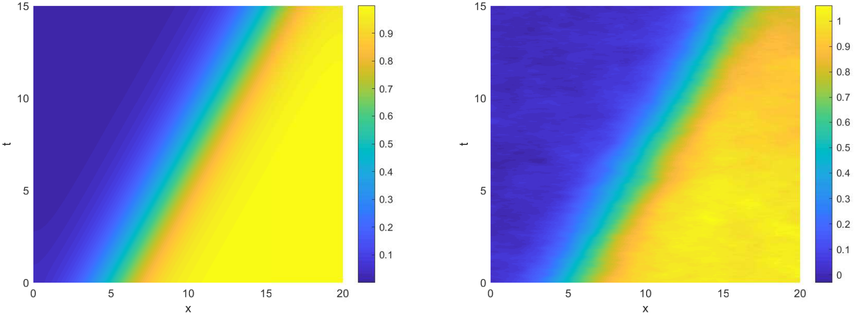
Motivation

Consider the Nagumo equation

$$\mathrm{d}x_t = [\Delta x_t + x_t(x_t - 1)(a - x_t)]\,\mathrm{d}t + \sigma \mathrm{d}W_t, \quad x_0 = x \in L^2(\Lambda),$$

with Neumann boundary conditions, where $\Lambda \subset \mathbb{R}$ is bounded and $a \in (0, 1)$.



(a) Deterministic Case, $\sigma = 0$

(b) Stochastic Case, $\sigma = 0.5$

Figure: Uncontrolled Nagumo Equation

Motivation

Fix $T > 0$. Introduce control $u : [0, T] \times \Lambda \times \Omega \to \mathbb{R}$

$$\mathrm{d}x_t^u = [\Delta x_t^u + x_t^u(x_t^u - 1)(a - x_t^u) + u_t]\,\mathrm{d}t + \sigma \mathrm{d}W_t, \quad x_0^u = x \in L^2(\Lambda), \tag{1}$$

and cost functional

$$\begin{aligned} J(u) :=& \mathbb{E}\Bigg[\int_0^T \int_\Lambda \frac{c_{\overline{\Lambda}}}{2} \left(x_t^u(\lambda) - x_{\overline{\Lambda}}(t,\lambda)\right)^2 + \frac{\nu}{2} u^2(t,\lambda)\,\mathrm{d}\lambda \mathrm{d}t \\ & + \int_\Lambda \frac{c_T}{2} \left(x_T^u(\lambda) - x^T(\lambda)\right)^2 \mathrm{d}\lambda \Bigg], \end{aligned}$$

where $c_{\overline{\Lambda}}, \nu, c_T \geq 0$.
Goal:

$$\boxed{\text{Minimize} \quad J \quad \text{subject to (1)}}$$

Outline

Setting

Minimize

$$J(u) := \mathbb{E}\left[\int_0^T \int_\Lambda l(x_t^u(\lambda), u_t)\mathrm{d}\lambda\mathrm{d}t + \int_\Lambda h(x_T^u(\lambda))\mathrm{d}\lambda\right]$$

subject to

$$\begin{cases} \mathrm{d}x_t^u = [\Delta x_t^u + b(x_t^u, u_t)]\mathrm{d}t + \sigma(x_t^u, u_t)\mathrm{d}W_t, \quad t \in [0, T] \\ x_0^u = x \in L^2(\Lambda), \end{cases}$$

where

- b, σ Nemytskii operators, Lipschitz;
- $(W_t)_{t\geq 0}$ cylindrical Wiener process;
- $\Lambda \subset \mathbb{R}$ bounded interval;
- l, h Nemytskii operators of (at most) quadratic growth.

Goal: Derive necessary conditions for optimality.

Setting

Minimize

$$J(u) := \mathbb{E}\left[\int_0^T \int_\Lambda l(x_t^u(\lambda), u_t)\mathrm{d}\lambda\mathrm{d}t + \int_\Lambda h(x_T^u(\lambda))\mathrm{d}\lambda\right]$$

subject to

$$\begin{cases}\mathrm{d}x_t^u = [\Delta x_t^u + b(x_t^u, u_t)]\mathrm{d}t + \sigma(x_t^u, u_t)\mathrm{d}W_t, \quad t \in [0,T] \\ x_0^u = x \in L^2(\Lambda).\end{cases}$$

- Pontryagin's maximum principle (1956): Necessary optimality condition in finite-dimensional, deterministic case ($\sigma \equiv 0$).
- Peng's maximum principle (1990): Generalization to finite-dimensional stochastic case.
- Since then: Many generalizations to infinite-dimensional, stochastic case by Du, Fuhrman, Frankowska, Guatteri, Hu, Li, Lü, Meng, Tang, Tessitore, Zhang, ...
  Major drawback in previous works: Strong assumptions on coefficients l and h, in particular excluding quadratic costs.

Spike Variation

Let $\bar{u}$ be optimal. Fix $\tau \in [0, T)$, $\varepsilon > 0$, $v \in U$, and set

$$u_t^\varepsilon := \begin{cases} v, & \tau \leq t \leq \tau + \varepsilon, \\ \bar{u}_t & \text{otherwise.} \end{cases}$$

Then

$$0 \leq J(u^\varepsilon) - J(\bar{u}) = \mathbb{E}\left[\int_0^T \int_\Lambda l(x_t^\varepsilon, u_t^\varepsilon) - l(\bar{x}_t, \bar{u}_t)\mathrm{d}\lambda\mathrm{d}t + \int_\Lambda h(x_T^\varepsilon) - h(\bar{x}_T)\mathrm{d}\lambda\right].$$

Taylor expansions: For terminal costs we have

$$\mathbb{E}\left[\int_\Lambda h(x_T^\varepsilon) - h(\bar{x}_T)\mathrm{d}\lambda\right] \approx \mathbb{E}\left[\int_\Lambda h_x(\bar{x}_T)y_T^\varepsilon + \frac{1}{2}h_{xx}(\bar{x}_T)y_T^\varepsilon y_T^\varepsilon \mathrm{d}\lambda\right],$$

where y^ε satisfies linearized state equation

$$\begin{cases} \mathrm{d}y_t^\varepsilon = [\Delta y_t^\varepsilon + b_x(\bar{x}_t, \bar{u}_t)y_t^\varepsilon + b(\bar{x}_t, u_t^\varepsilon) - b(\bar{x}_t, \bar{u}_t)]\,\mathrm{d}t \\ \qquad\qquad + [\sigma_x(\bar{x}_t, \bar{u}_t)y_t^\varepsilon + \sigma(\bar{x}_t, u_t^\varepsilon) - \sigma(\bar{x}_t, \bar{u}_t)]\,\mathrm{d}W_t \\ y_0^\varepsilon = 0. \end{cases}$$

First Order Adjoint State

Consider SPDE

$$\begin{cases} \mathrm{d}y_t^\varepsilon = [\Delta y_t^\varepsilon + b_x(\bar{x}_t, \bar{u}_t) y_t^\varepsilon + \varphi_t]\,\mathrm{d}t + [\sigma_x(\bar{x}_t, \bar{u}_t) y_t^\varepsilon + \psi_t]\,\mathrm{d}W_t \\ y_0^\varepsilon = 0, \end{cases}$$

where $(\varphi, \psi) \in L^2([0,T] \times \Omega; L^2(\Lambda)) \times L^2([0,T] \times \Omega; L_2(\Xi, L^2(\Lambda)))$.
Construct linear functional

$$\mathcal{T}(\varphi, \psi) := \mathbb{E}\left[\int_0^T \int_\Lambda l_x(\bar{x}_t(\lambda), \bar{u}_t) y_t^\varepsilon(\lambda) \mathrm{d}\lambda \mathrm{d}t + \int_\Lambda h_x(\bar{x}_T(\lambda)) y_T^\varepsilon(\lambda) \mathrm{d}\lambda\right].$$

By Riesz's representation theorem, there is a unique pair

$$(p, q) \in L^2([0,T] \times \Omega; L^2(\Lambda)) \times L^2([0,T] \times \Omega; L_2(\Xi, L^2(\Lambda)))$$

such that

$$\mathcal{T}(\varphi, \psi) = \mathbb{E}\left[\int_0^T \langle \varphi_t, p_t \rangle_{L^2(\Lambda)} + \langle \psi_t, q_t \rangle_{L_2(\Xi, L^2(\Lambda))} \mathrm{d}t\right]$$

for all $(\varphi, \psi) \in L^2([0,T] \times \Omega; L^2(\Lambda)) \times L^2([0,T] \times \Omega; L_2(\Xi, L^2(\Lambda)))$.

First Order Adjoint Equation

Adjoint state property:

$$\mathbb{E}\left[\int_0^T \langle y_t, l_x(\bar{x}_t, \bar{u}_t)\rangle \mathrm{d}t + \langle h_x(\bar{x}_T), y_T\rangle\right] = \mathbb{E}\left[\int_0^T \langle p_t, \varphi_t\rangle + \langle q_t, \psi_t\rangle \mathrm{d}t\right].$$

Applying Itô's product rule yields

$$\begin{aligned}\mathrm{d}\langle p_t, y_t\rangle_{L^2(\Lambda)} =& \langle p_t, \mathrm{d}y_t\rangle_{L^2(\Lambda)} + \langle y_t, \mathrm{d}p_t\rangle_{L^2(\Lambda)} + \mathrm{d}\langle p, y\rangle_t \\ \overset{!}{=}& \left[\langle p_t, \varphi_t\rangle_{L^2(\Lambda)} + \langle q_t, \psi_t\rangle_{L_2(\Xi, L^2(\Lambda))} - \langle y_t, l_x(\bar{x}_t, \bar{u}_t)\rangle_{L^2(\Lambda)}\right] \mathrm{d}t + dM_t.\end{aligned}$$

for some martingale $(M_t)_{t\geq 0}$. Thus,

$$\begin{cases}\mathrm{d}p_t = -\left[\Delta p_t + b_x(\bar{x}_t, \bar{u}_t)p_t + l_x(\bar{x}_t, \bar{u}_t) + \langle \sigma_x(\bar{x}_t, \bar{u}_t), q_t\rangle_{L_2(\Xi, \mathbb{R})}\right] \mathrm{d}t + q_t \mathrm{d}W_t \\ p_T = h_x(\bar{x}_T).\end{cases}$$

Unique variational solution (p, q), where

$$p \in L^2([0, T] \times \Omega; H_0^1(\Lambda)) \cap L^2(\Omega; C([0, T]; L^2(\Lambda)))$$

and

$$q \in L^2([0, T] \times \Omega; L_2(\Xi, L^2(\Lambda))).$$

Backward Stochastic Differential Equations

Let $(W_t)_{t\geq 0}$ be a Brownian motion and $(\mathcal{F}_t)_{t\geq 0}$ its natural filtration. Consider the BSDE

$$\begin{cases} \mathrm{d}p_t = 0 \\ p_T = \xi, \end{cases}$$

where ξ is $\mathcal{F}_T$-measurable. ⚠ Adapted solution doesn't exist!
Natural candidate: $p_t = \mathbb{E}[\xi|\mathcal{F}_t]$. However, does not solve BSDE. Martingale representation theorem:

$$\mathbb{E}[\xi|\mathcal{F}_t] = \mathbb{E}[\xi] + \int_0^t q_s \mathrm{d}W_s.$$

Restate original problem:

$$\begin{cases} \mathrm{d}p_t = q_t \mathrm{d}W_t \\ p_T = \xi. \end{cases}$$

For $\xi = f(W_T)$:

$$\mathbb{E}\left[\xi|\mathcal{F}_t\right] = \mathbb{E}\left[f(W_T)|\mathcal{F}_t\right] = S_{T-t}f(W_t) = S_T f(W_0) + \int_0^t \nabla S_{T-s} f(W_s)\mathrm{d}W_s.$$

Quadratic Terms

Taylor expanding the cost functional to second order leads to the quadratic terms

$$\mathbb{E}\left[\int_0^T \int_\Lambda l_{xx}(\bar{x}_t(\lambda), \bar{u}_t) y_t^\varepsilon(\lambda) y_t^\varepsilon(\lambda) \mathrm{d}\lambda \mathrm{d}t + \int_\Lambda h_{xx}(\bar{x}_T(\lambda)) y_T^\varepsilon(\lambda) y_T^\varepsilon(\lambda) \mathrm{d}\lambda\right].$$

Idea: Linearize using tensor product. In finite dimensions, Peng derived equation for $y_t^\varepsilon \otimes y_t^\varepsilon$ on

$$\mathbb{R}^n \otimes \mathbb{R}^n \cong \mathbb{R}^{n \times n}.$$

In existing literature, this is generalized in infinite dimensions to

$$H \otimes H \cong L_2(H).$$

Problem: In order to perform duality analysis, we need to solve equation in $L_1(H)$.

Explicit Tensor Product

Instead, we use explicit representation

$$\begin{aligned} L^2(\Lambda) \otimes L^2(\Lambda) &\cong L^2(\Lambda^2) \\ y \otimes z &\leftrightsquigarrow ((\lambda, \mu) \mapsto y(\lambda) z(\mu)) \,. \end{aligned}$$

Thus, we can rewrite quadratic terms as

$$\begin{aligned} &\mathbb{E}\left[\int_0^T \int_\Lambda l_{xx}(\bar{x}_t(\lambda), \bar{u}_t) y_t^\varepsilon(\lambda) y_t^\varepsilon(\lambda) \mathrm{d}\lambda \mathrm{d}t\right] \\ =&\mathbb{E}\left[\int_0^T \int_\Lambda l_{xx}(\bar{x}_t(\lambda), \bar{u}_t) \delta(y_t^\varepsilon \otimes y_t^\varepsilon)(\lambda) \mathrm{d}\lambda \mathrm{d}t\right], \end{aligned}$$

where $\delta : H_0^1(\Lambda^2) \to L^2(\Lambda)$ is defined by $\delta(w)(\lambda) := w(\lambda, \lambda)$. This expression is linear in $y_t^\varepsilon \otimes y_t^\varepsilon$.

Second Order Adjoint Equation

Theorem (Stannat, W., SICON 2021)

The equation

$$\begin{cases} \mathrm{d}P_t(\lambda,\mu) = -[\Delta P_t(\lambda,\mu) + (b_x(\bar{x}_t(\lambda),\bar{u}_t) + b_x(\bar{x}_t(\mu),\bar{u}_t))P_t(\lambda,\mu) \\ \qquad +\langle \sigma_x(\bar{x}_t(\lambda),\bar{u}_t), \sigma_x(\bar{x}_t(\mu),\bar{u}_t)\rangle_{L_2(\Xi,\mathbb{R})} P_t(\lambda,\mu) \\ \qquad +\langle \sigma_x(\bar{x}_t(\lambda),\bar{u}_t) + \sigma_x(\bar{x}_t(\mu),\bar{u}_t), Q_t(\lambda,\mu)\rangle_{L_2(\Xi,\mathbb{R})} \\ \qquad +\delta^*(l_{xx}(\bar{x}_t(\lambda),\bar{u}_t)) + \delta^*(b_{xx}(\bar{x}_t(\lambda),\bar{u}_t)p_t(\lambda)) \\ \qquad +\delta^*(\langle \sigma_{xx}(\bar{x}_t(\lambda),\bar{u}_t), q_t\rangle_{L_2(\Xi,\mathbb{R})})]\mathrm{d}t + Q_t(\lambda,\mu)\mathrm{d}W_t \\ P_T(\lambda,\mu) = \delta^*(h_{xx}(\bar{x}_T(\lambda))) \end{cases}$$

has a unique adapted solution (P,Q), *where*

$$P \in L^2([0,T]\times\Omega; L^2(\Lambda^2)) \cap L^2(\Omega; C([0,T]; H^{-1}(\Lambda^2))),$$

and

$$Q \in L^2([0,T]\times\Omega; L_2(\Xi; H^{-1}(\Lambda^2))).$$

Peng's Maximum Principle for SPDEs

Theorem (Stannat, W., SICON 2021)

Let $(\bar{x}, \bar{u})$ be an optimal pair. Then there exist adapted processes

$$(p, q) \in L^2([0,T] \times \Omega; H_0^1(\Lambda)) \times L^2([0,T] \times \Omega; L_2(\Xi, L^2(\Lambda)))$$

satisfying the first order adjoint equation and adapted processes

$$(P, Q) \in L^2([0,T] \times \Omega; L^2(\Lambda^2)) \times L^2([0,T] \times \Omega; L_2(\Xi, H^{-1}(\Lambda^2)))$$

satisfying the second order adjoint equation such that

$$\inf_{u \in U} \mathcal{G}(t, \bar{x}_t, u) = \mathcal{G}(t, \bar{x}_t, \bar{u}_t),$$

for almost all $(t, \omega) \in [0,T] \times \Omega$, where $\mathcal{G} : [0,T] \times L^2(\Lambda) \times U \to \mathbb{R}$

$$\mathcal{G}(t, x, u) := \int_\Lambda l(x(\lambda), u) \mathrm{d}\lambda + \langle p_t, b(x,u) \rangle_{L^2(\Lambda)} + \frac{1}{2} tr(\sigma(x,u)^* P_t \sigma(x,u)) + tr\left(\sigma(x,u)^* \left[q_t - P_t \sigma(\bar{x}_t, \bar{u}_t)\right]\right).$$

Gradient of the Cost Functional

Minimize

$$J(u)=\mathbb{E}\left[\int_0^T\int_\Lambda \frac{c_{\overline{\Lambda}}}{2}\left(x_t^u(\lambda)-x_{\overline{\Lambda}}(t,\lambda)\right)^2+\frac{\nu}{2}u^2(t,\lambda)\mathrm{d}\lambda\mathrm{d}t+\int_\Lambda \frac{c_T}{2}\left(x_T^u(\lambda)-x^T(\lambda)\right)^2\mathrm{d}\lambda\right]$$

over $u \in L^2([0,T]\times\Lambda)$ subject to

$$\mathrm{d}x_t^u = \left[\Delta x_t^u + b(x_t^u) + u_t\right]\mathrm{d}t + \sigma \mathrm{d}W_t, \quad x_0^u = x \in L^2(\Lambda),$$

where $b(x) = x(x-1)(a-x)$.
Major changes:

- Additive noise;
- Deterministic controls;
- Nonlinearity not Lipschitz.

Gradient of the Cost Functional

Minimize

$$J(u)=\mathbb{E}\left[\int_0^T\int_\Lambda \frac{c_{\overline{\Lambda}}}{2}\left(x_t^u(\lambda)-x_{\overline{\Lambda}}(t,\lambda)\right)^2+\frac{\nu}{2}u^2(t,\lambda)\mathrm{d}\lambda\mathrm{d}t+\int_\Lambda \frac{c_T}{2}\left(x_T^u(\lambda)-x^T(\lambda)\right)^2\mathrm{d}\lambda\right]$$

over $u \in L^2([0,T]\times\Lambda)$ subject to

$$\mathrm{d}x_t^u = \left[\Delta x_t^u + b(x_t^u) + u_t\right]\mathrm{d}t + \sigma \mathrm{d}W_t, \quad x_0^u = x \in L^2(\Lambda),$$

where $b(x) = x(x-1)(a-x)$.

Theorem (Stannat, W., EECT 2021)

The gradient of the cost functional is given by

$$\nabla J(u)(t,\lambda) = \mathbb{E}\left[p_t(\lambda) + \nu u(t,\lambda)\right],$$

where p is the solution of the adjoint equation

$$\begin{cases} -\partial_t p_t = \Delta p_t + b'(x_t^u)p_t + c_{\overline{\Lambda}}\left(x_t^u - x_{\overline{\Lambda}}(t,\cdot)\right), & t\in[0,T] \\ p_T = c_T\left(x_T^u - x^T\right) \in L^2(\Lambda). \end{cases}$$

Gradient Descent Algorithm

Fix initial control $u_0 \in L^6([0, T] \times \Lambda)$, step size $s_0 > 0$ and stopping criterion $\eta > 0$.

1. Solve state equation.
2. Solve adjoint equation.
3. Repeat Step 1 and Step 2 to approximate

$$\nabla J(u_n)(t, \lambda) = \mathbb{E}\left[p_t^n(\lambda) + \nu u_n(t, \lambda)\right]$$

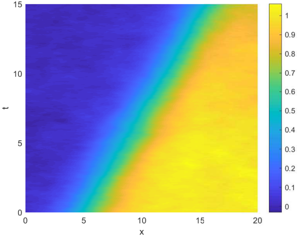
   via a Monte Carlo method.
4. Direction of descent: $D_n = -\nabla J(u_n) + \beta_n D_{n-1}$, where $\beta_n = \frac{\|\nabla J(u_n)\|}{\|\nabla J(u_{n-1})\|}$.
5. Compute the new control via $u_{n+1} = u_n + s_n D_n$.
6. Stop if $\|\nabla J(u_n)\| < \eta$, otherwise reset the step size $s_n = s_0$ and go to step 1.

Simulations

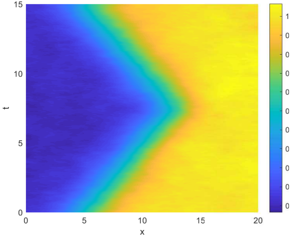
Consider

$$\begin{cases} \mathrm{d}x_t^u = [\Delta x_t^u + x_t^u(x_t^u - 1)(a - x_t^u) + u_t]\,\mathrm{d}t + \sigma \mathrm{d}W_t, & t \in [0, T] \\ x_0^u = x \in L^2(\Lambda), \end{cases}$$

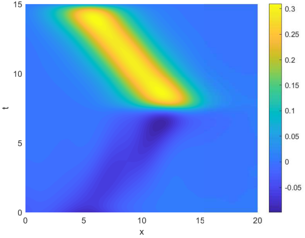
where $a = 39/40$, $\sigma = 1/2$, $T = 15$, $\Lambda = [0, 20]$.



(a) Uncontrolled Solution



(b) Controlled Solution



(c) Optimal Control

Outlook

Consider feedback control $u_t = \mathfrak{u}(t, x_t^u)$. Leads to

$$\mathrm{d}x_t^{\mathfrak{u}} = [\Delta x_t^{\mathfrak{u}} + x_t^{\mathfrak{u}}(x_t^{\mathfrak{u}} - 1)(a - x_t^{\mathfrak{u}}) + \mathfrak{u}(t, x_t^{\mathfrak{u}})]\,\mathrm{d}t + \sigma \mathrm{d}W_t, \quad t \in [0, T].$$

Questions:

- Existence of solution?
- How to approximate $\mathfrak{u}$?

Restrict class of controls to

$$\Phi(t, x, u) = \sum_{i=1}^{N} u^i(t, \cdot) \exp\left(-\|x_t - v^i\|_{L^2(\Lambda)}^2\right).$$

Optimize over *deterministic* u^i, v^i using our algorithm.

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