Optimal Control of Stochastic Reaction-Diffusion Equations

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Stochastic Reaction-Diffusion Equations

Consider

$$\partial_t x_t(\lambda) = \Delta x_t(\lambda) + b(x_t(\lambda)) + \xi_t(\lambda), \quad \lambda \in \Lambda \subset \mathbb{R}^n,$$

where

- Δ – Laplace operator
- $b : \mathbb{R} \to \mathbb{R}$ local reaction term
- ξ – random fluctuations.

Formulate as stochastic integral equation

$$\mathrm{d}x_t = [\Delta x_t + b(x_t)]\,\mathrm{d}t + \sigma \mathrm{d}W_t \quad \text{in } L^2(\Lambda),$$

where

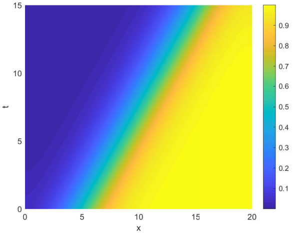
- σ – noise intensity
- $(W_t)_t$ cylindrical Wiener process.

Example

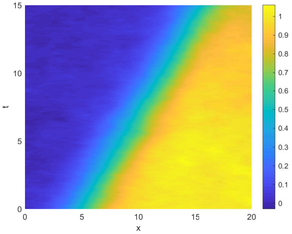
Consider the Nagumo equation

$$\mathrm{d}x_t = [\Delta x_t + x_t(x_t - 1)(a - x_t)]\,\mathrm{d}t + \sigma \mathrm{d}W_t, \quad x_0 = x \in L^2(\Lambda),$$

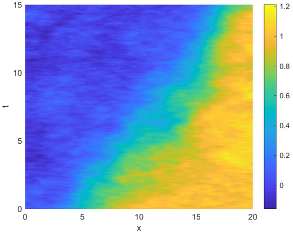
with Neumann boundary conditions, where $\Lambda \subset \mathbb{R}$ is bounded and $a \in (0, 1)$.



$\sigma = 0$



$\sigma = \frac{1}{2}$



$\sigma = 2$

Control of the Stochastic Nagumo Equation

Fix $T > 0$. Introduce control $u : [0, T] \times \Lambda \times \Omega \to \mathbb{R}$

$$\begin{cases} \mathrm{d}x_t^u = [\Delta x_t^u + x_t^u(x_t^u - 1)(a - x_t^u) + u_t]\,\mathrm{d}t + \sigma \mathrm{d}W_t \\ x_0^u = x \in L^2(\Lambda), \end{cases} \qquad (\star)$$

and cost functional

$$J(u) = \mathbb{E}\left[\int_0^T \int_\Lambda \frac{c_{\overline{\Lambda}}}{2} \left(x_t^u(\lambda) - x_{\overline{\Lambda}}(t,\lambda)\right)^2 + \frac{\nu}{2} u^2(t,\lambda)\mathrm{d}\lambda \mathrm{d}t + \int_\Lambda \frac{c_T}{2} \left(x_T^u(\lambda) - x^T(\lambda)\right)^2 \mathrm{d}\lambda\right]$$

where

- $x_{\overline{\Lambda}}$, x^T – desired reference profiles
- $c_{\overline{\Lambda}}, \nu, c_T \geq 0$.

Goal:

$$\boxed{\text{Minimize} \quad J \quad \text{subject to } (\star)}$$

Outline

1. Necessary Optimality Conditions
2. Sufficient Optimality Conditions
3. Applications

Setting

Minimize

$$J(u) := \mathbb{E}\left[\int_0^T \int_\Lambda l(x_t^u(\lambda), u_t)\mathrm{d}\lambda\mathrm{d}t + \int_\Lambda h(x_T^u(\lambda))\mathrm{d}\lambda\right]$$

subject to

$$\begin{cases} \mathrm{d}x_t^u = [\Delta x_t^u + b(x_t^u, u_t)]\mathrm{d}t + \sigma(x_t^u, u_t)\mathrm{d}W_t, \quad t \in [0, T] \\ x_0^u = x \in L^2(\Lambda), \end{cases}$$

where

- l, h Nemytskii operators of (at most) quadratic growth
- b, σ Nemytskii operators, Lipschitz
- $\Lambda \subset \mathbb{R}$ bounded interval (also extensions to higher dimensions)
- control domain U non-convex.

Setting

Minimize

$$J(u) := \mathbb{E}\left[\int_0^T \int_\Lambda l(x_t^u(\lambda), u_t)\mathrm{d}\lambda\mathrm{d}t + \int_\Lambda h(x_T^u(\lambda))\mathrm{d}\lambda\right]$$

subject to

$$\begin{cases} \mathrm{d}x_t^u = [\Delta x_t^u + b(x_t^u, u_t)]\mathrm{d}t + \sigma(x_t^u, u_t)\mathrm{d}W_t, \quad t \in [0, T] \\ x_0^u = x \in L^2(\Lambda). \end{cases}$$

- Pontryagin's maximum principle (1956): Necessary optimality condition for controlled ODEs.
- Peng's maximum principle (1990): Generalization to controlled SDEs.
- Since then: Generalizations to controlled SPDEs by Du, Fuhrman, Frankowska, Guatteri, Hu, Li, Lü, Meng, Tang, Tessitore, Zhang, ...
  Previous results require strong assumptions on coefficients l and h excluding quadratic costs.

Spike Variation

Let $\bar{u}$ be optimal. Fix $\tau \in [0, T)$, $\varepsilon > 0$, $v \in U$, and set

$$u_t^\varepsilon := \begin{cases} v, & \tau \leq t \leq \tau + \varepsilon, \\ \bar{u}_t & \text{otherwise.} \end{cases}$$

Then

$$0 \leq J(u^\varepsilon) - J(\bar{u}) = \mathbb{E}\left[\int_0^T \int_\Lambda l(x_t^\varepsilon, u_t^\varepsilon) - l(\bar{x}_t, \bar{u}_t)\mathrm{d}\lambda\mathrm{d}t + \int_\Lambda h(x_T^\varepsilon) - h(\bar{x}_T)\mathrm{d}\lambda\right].$$

Roadmap: 1) Taylor expand integrands.
2) Divide by ε, send $\varepsilon \to 0$.
3) Identify remaining terms.

Because of stochastic calculus, we have to Taylor expand up to second order.

Use representation

$$L^2(\Lambda) \otimes L^2(\Lambda) \cong L^2(\Lambda^2).$$

Second Order Adjoint Equation

Theorem (Stannat, W., SICON 2021)

The equation

$$\begin{cases} \mathrm{d}P_t(\lambda,\mu) = -[\Delta P_t(\lambda,\mu) + (b_x(\bar{x}_t(\lambda),\bar{u}_t) + b_x(\bar{x}_t(\mu),\bar{u}_t))P_t(\lambda,\mu) \\ \qquad +\langle \sigma_x(\bar{x}_t(\lambda),\bar{u}_t), \sigma_x(\bar{x}_t(\mu),\bar{u}_t)\rangle_{L_2(\Xi,\mathbb{R})} P_t(\lambda,\mu) \\ \qquad +\langle \sigma_x(\bar{x}_t(\lambda),\bar{u}_t) + \sigma_x(\bar{x}_t(\mu),\bar{u}_t), Q_t(\lambda,\mu)\rangle_{L_2(\Xi,\mathbb{R})} \\ \qquad +\delta^*(l_{xx}(\bar{x}_t(\lambda),\bar{u}_t)) + \delta^*(b_{xx}(\bar{x}_t(\lambda),\bar{u}_t)p_t(\lambda)) \\ \qquad +\delta^*(\langle \sigma_{xx}(\bar{x}_t(\lambda),\bar{u}_t), q_t\rangle_{L_2(\Xi,\mathbb{R})})]\mathrm{d}t + Q_t(\lambda,\mu)\mathrm{d}W_t \\ P_T(\lambda,\mu) = \delta^*(h_{xx}(\bar{x}_T(\lambda))) \end{cases}$$

has a unique adapted solution (P,Q), *where*

$$P \in L^2([0,T]\times\Omega; L^2(\Lambda^2)) \cap L^2(\Omega; C([0,T]; H^{-1}(\Lambda^2))),$$

and

$$Q \in L^2([0,T]\times\Omega; L_2(\Xi; H^{-1}(\Lambda^2))).$$

Peng's Maximum Principle for SPDEs

Theorem (Stannat, W., SICON 2021)

Let $(\bar{x}, \bar{u})$ *be an optimal pair. Then there exist adapted processes*

$$(p, q) \in L^2([0, T] \times \Omega; H_0^1(\Lambda)) \times L^2([0, T] \times \Omega; L_2(\Xi, L^2(\Lambda)))$$

satisfying the first order adjoint equation and adapted processes

$$(P, Q) \in L^2([0, T] \times \Omega; L^2(\Lambda^2)) \times L^2([0, T] \times \Omega; L_2(\Xi, H^{-1}(\Lambda^2)))$$

satisfying the second order adjoint equation such that

$$\inf_{u \in U} \mathcal{G}(t, \bar{x}_t, u) = \mathcal{G}(t, \bar{x}_t, \bar{u}_t),$$

for almost all $(t, \omega) \in [0, T] \times \Omega$*, where* $\mathcal{G} : [0, T] \times L^2(\Lambda) \times U \to \mathbb{R}$

$$\mathcal{G}(t, x, u) := \int_\Lambda l(x(\lambda), u) \mathrm{d}\lambda + \langle p_t, b(x, u) \rangle_{L^2(\Lambda)} + tr(\sigma(x, u)^* q_t) + \frac{1}{2} tr(\sigma(x, u)^* P_t \sigma(x, u)) - tr(\sigma(x, u)^* P_t \sigma(\bar{x}_t, \bar{u}_t)).$$

Dynamic Programming Approach

Minimize

$$J(s,x;u) := \mathbb{E}\left[\int_s^T \int_\Lambda l(x_t^u(\lambda), u_t)\mathrm{d}\lambda\mathrm{d}t + \int_\Lambda h(x_T^u(\lambda))\mathrm{d}\lambda\right]$$

over $u \in \mathcal{U}_s$ subject to

$$\begin{cases} \mathrm{d}x_t^u = [\Delta x_t^u + b(x_t^u, u_t)]\mathrm{d}t + \sigma(x_t^u, u_t)\mathrm{d}W_t, & t \in [s, T] \\ x_s^u = x \in L^2(\Lambda). \end{cases}$$

Introduce value function $V : [0, T] \times L^2(\Lambda) \to \mathbb{R}$,

$$V(s,x) := \inf_{u \in \mathcal{U}_s} J(s,x;u).$$

Satisfies dynamic programming principle

$$V(s,x) = \inf_{u \in \mathcal{U}_s} \mathbb{E}\left[\int_s^\tau l(x_t^u, u_t)\mathrm{d}t + V(\tau, x_\tau^u)\right], \qquad \forall \tau \in [s, T].$$

Dynamic Programming Approach

Minimize

$$J(s,x;u) := \mathbb{E}\left[\int_s^T \int_\Lambda l(x_t^u(\lambda), u_t)\mathrm{d}\lambda\mathrm{d}t + \int_\Lambda h(x_T^u(\lambda))\mathrm{d}\lambda\right]$$

over $u \in \mathcal{U}_s$ subject to

$$\begin{cases} \mathrm{d}x_t^u = [\Delta x_t^u + b(x_t^u, u_t)]\mathrm{d}t + \sigma(x_t^u, u_t)\mathrm{d}W_t, \quad t \in [s, T] \\ x_s^u = x \in L^2(\Lambda). \end{cases}$$

Value function formally satisfies Hamilton-Jacobi-Bellman equation

$$\begin{cases} V_s + \langle \Delta x, DV\rangle_{L^2(\Lambda)} + \inf_{u\in U} \mathcal{H}(x,u,DV,D^2V) = 0, \quad (s,x) \in (0,T) \times L^2(\Lambda) \\ V(T,x) = \int_\Lambda h(x(\lambda))\mathrm{d}\lambda, \quad x \in L^2(\Lambda) \end{cases}$$

where

$$\mathcal{H}(x,u,p,P) := \int_\Lambda l(x(\lambda),u)\mathrm{d}\lambda + \langle p, b(x,u)\rangle_{L^2(\Lambda)} + \frac{1}{2}\mathrm{tr}\left(\sigma(x,u)^* P\sigma(x,u)\right).$$

Can be used to derive optimality conditions.

Link Between Value Function and Adjoint States

Under smoothness assumptions, it holds

$$\begin{cases} V_s(t,\bar{x}_t) = -\langle \Delta \bar{x}_t, DV(t,\bar{x}_t)\rangle_{L^2(\Lambda)} - \mathcal{H}(\bar{x}_t, \bar{u}_t, DV(t,\bar{x}_t), D^2V(t,\bar{x}_t)) \\ DV(t,\bar{x}_t) = p_t \\ D^2V(t,\bar{x}_t)\sigma(\bar{x}_t,\bar{u}_t) = q_t. \end{cases}$$

Generalizations dropping smoothness assumptions and using viscosity differentials up to first order in infinite dimensions by Cannarsa, Frankowska, Zhou, ...

Theorem (Stannat, W. (2022+))

For almost every $t \in [0,T]$, it holds that

$$[-\langle \Delta \bar{x}_t, p_t\rangle_{H^{-1}(\Lambda)\times H_0^1(\Lambda)} - \mathcal{G}(t,\bar{x}_t,\bar{u}_t), \infty) \times \{p_t\} \times [P_t, \infty) \subset D^{1,2,+}_{t+,x} V(t,\bar{x}_t)$$

$\mathbb{P}$-almost surely.

Link Between Value Function and Adjoint States

Under smoothness assumptions, it holds

$$\begin{cases} V_s(t,\bar{x}_t) = -\langle \Delta \bar{x}_t, DV(t,\bar{x}_t)\rangle_{L^2(\Lambda)} - \mathcal{H}(\bar{x}_t, \bar{u}_t, DV(t,\bar{x}_t), D^2V(t,\bar{x}_t)) \\ DV(t,\bar{x}_t) = p_t \\ D^2V(t,\bar{x}_t)\sigma(\bar{x}_t,\bar{u}_t) = q_t. \end{cases}$$

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Corollary (Stannat, W. (2022+))

It holds for almost all $t \in [0, T]$

$$\mathcal{G}(t,\bar{x}_t,\bar{u}_t) \le \mathcal{H}(t,\bar{x}_t,\bar{u}_t,p_t,P_t),$$

$\mathbb{P}$*-almost surely, i.e.,*

$$tr(\sigma(\bar{x}_t,\bar{u}_t)(q_t - P_t\sigma(\bar{x}_t,\bar{u}_t))) \le 0.$$

Verification Theorem

Minimize

$$J(s,x;u) := \mathbb{E}\left[\int_s^T L(x_t^u, u_t)\mathrm{d}t + H(x_T^u)\right]$$

over $u \in \mathcal{U}_s$ subject to

$$\begin{cases} \mathrm{d}x_t^u = [\Delta x_t^u + B(x_t^u, u_t)]\mathrm{d}t + \Sigma(x_t^u, u_t)\mathrm{d}W_t, & t \in [s, T] \\ x_s^u = x \in L^2(\Lambda). \end{cases}$$

Smooth verification theorem: Let (x^*, u^*) be admissible such that

$$u_t^* \in \arg\min_{u \in U} \mathcal{H}(x_t^*, u, DV(t, x_t^*), D^2V(t, x_t^*)),$$

for almost every (t, ω). Then (x^*, u^*) is optimal.

Verification theorems in the stochastic case in the framework of viscosity solutions by Fabbri, Gozzi, Li, Święch, Yong, Zhou, ...

Verification Theorem

Minimize

$$J(s, x; u) := \mathbb{E}\left[\int_s^T L(x_t^u, u_t)\mathrm{d}t + H(x_T^u)\right]$$

over $u \in \mathcal{U}_s$ subject to

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Smooth verification theorem: Let (x^*, u^*) be admissible such that

$$V_s(t, x_t^*) + \langle \Delta x_t^*, DV(t, x_t^*)\rangle_{L^2(\Lambda)} + \mathcal{H}(x^*, u_t^*, DV(t, x_t^*), D^2V(t, x_t^*)) = 0$$

for almost every (t, ω). Then (x^*, u^*) is optimal.

Verification theorems in the stochastic case in the framework of viscosity solutions by Fabbri, Gozzi, Li, Święch, Yong, Zhou, ...

Verification Theorem

Theorem (Stannat, W. (2022+))

Assume

- $\|\Sigma(x,u)\|_{L_2(\Xi, H_0^1(\Lambda))} \leq C(1 + \|x\|_{H_0^1(\Lambda)})$
- $V(t+\tau, x) - V(t,x) \leq C(1 + \|x\|^2_{H_0^1(\Lambda)})\tau$
- $V(t,\cdot) - C\|\cdot\|^2_{L^2(\Lambda)}$ *is concave.*

Let (x^*, u^*) *be an admissible pair. Suppose there are adapted processes* (G, p, P) *taking values in* $\mathbb{R}$, $H_0^1(\Lambda)$ *and* $L_2(L^2(\Lambda))$*, such that for almost all* $t \in [s, T]$*:*

$$(G_t, p_t, P_t) \in D^{1,2,+}_{t+,x} V(t, x_t^*)$$

$\mathbb{P}$*-almost surely, and*

$$\mathbb{E}\left[\int_s^T G_t + \langle \Delta x_t^*, p_t \rangle_{H^{-1}(\Lambda)\times H_0^1(\Lambda)} + \mathcal{H}(x_t^*, u_t^*, p_t, P_t)\mathrm{d}t\right] \leq 0.$$

Then (x^*, u^*) *is optimal.*

Approximation of Optimal Controls

Numerical approximations by Dunst, Majee, Prohl, Vallet, ...

Minimize

$$J(u)=\mathbb{E}\left[\int_0^T\int_\Lambda \frac{c_{\overline{\Lambda}}}{2}\left(x_t^u(\lambda)-x_{\overline{\Lambda}}(t,\lambda)\right)^2+\frac{\nu}{2}u^2(t,\lambda)\mathrm{d}\lambda\mathrm{d}t+\int_\Lambda \frac{c_T}{2}\left(x_T^u(\lambda)-x^T(\lambda)\right)^2\mathrm{d}\lambda\right]$$

over $u \in L^2([0,T]\times\Lambda)$ subject to

$$\mathrm{d}x_t^u = \left[\Delta x_t^u + b(x_t^u) + u_t\right]\mathrm{d}t + \sigma \mathrm{d}W_t, \quad x_0^u = x \in L^2(\Lambda),$$

where $b(x) = x(x-1)(a-x)$.

Major changes:

- additive noise
- deterministic controls
- convex control domain
- nonlinearity not Lipschitz.

Approximation of Optimal Controls

Numerical approximations by Dunst, Majee, Prohl, Vallet, ...

Minimize

$$J(u) = \mathbb{E}\left[\int_0^T\int_\Lambda \frac{c_{\overline{\Lambda}}}{2}\left(x_t^u(\lambda) - x_{\overline{\Lambda}}(t,\lambda)\right)^2 + \frac{\nu}{2}u^2(t,\lambda)\mathrm{d}\lambda\mathrm{d}t + \int_\Lambda \frac{c_T}{2}\left(x_T^u(\lambda) - x^T(\lambda)\right)^2\mathrm{d}\lambda\right]$$

over $u \in L^2([0,T]\times\Lambda)$ subject to

$$\mathrm{d}x_t^u = \left[\Delta x_t^u + b(x_t^u) + u_t\right]\mathrm{d}t + \sigma\mathrm{d}W_t, \quad x_0^u = x \in L^2(\Lambda),$$

where $b(x) = x(x-1)(a-x)$.

Theorem (Stannat, W., EECT 2021)

The gradient of the cost functional is given by

$$\nabla J(u)(t,\lambda) = \mathbb{E}\left[p_t(\lambda) + \nu u(t,\lambda)\right],$$

where p is the solution of the adjoint equation

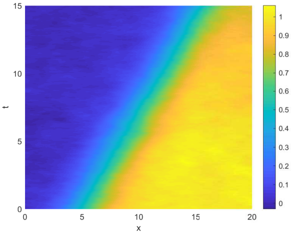
$$\begin{cases} -\partial_t p_t = \Delta p_t + b'(x_t^u)p_t + c_{\overline{\Lambda}}\left(x_t^u - x_{\overline{\Lambda}}(t,\cdot)\right), & t \in (0,T) \\ p_T = c_T\left(x_T^u - x^T\right) \in L^2(\Lambda). \end{cases}$$

Simulations

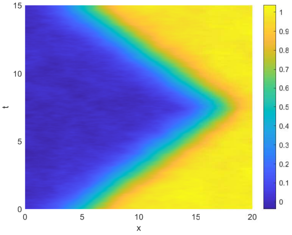
Consider

$$\begin{cases} \mathrm{d}x_t^u = [\Delta x_t^u + x_t^u(x_t^u - 1)(a - x_t^u) + u_t]\,\mathrm{d}t + \sigma \mathrm{d}W_t, & t \in [0, T] \\ x_0^u = x \in L^2(\Lambda), \end{cases}$$

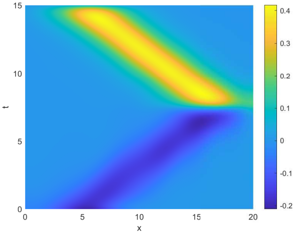
where $a = 39/40$, $\sigma = 1/2$, $T = 15$, $\Lambda = [0, 20]$.



Uncontrolled Solution



Controlled Solution



Optimal Control

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