Peng's Maximum Principle for Stochastic Partial Differential Equations

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Setting

Minimize

$$J(u) := \mathbb{E}\left[\int_0^T \int_\Lambda l(x_t^u(\lambda), u_t)\mathrm{d}\lambda\mathrm{d}t + \int_\Lambda h(x_T^u(\lambda))\mathrm{d}\lambda\right]$$

subject to

$$\begin{cases} \mathrm{d}x_t^u = [\Delta x_t^u + b(x_t^u, u_t)]\mathrm{d}t + \sigma(x_t^u, u_t)\mathrm{d}W_t & \text{on } [0, T] \times L^2(\Lambda) \\ x_0^u = x_0 & \text{in } L^2(\Lambda), \end{cases}$$

where

- b, σ Nemytskii operators, Lipschitz;
- $(W_t)_{t \geq 0}$ cylindrical Wiener process;
- $\Lambda \subset \mathbb{R}$ bounded interval;
- l, h Nemytskii operators of (at most) quadratic growth.

Goal: Derive necessary condition for optimality.

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- Pontryagin's maximum principle (1956): Necessary optimality condition in finite-dimensional, deterministic case ($\sigma \equiv 0$).
- Peng's maximum principle (1990): Generalization to finite-dimensional stochastic case.
- Since then: Many generalizations to infinite-dimensional stochastic case by Du, Fuhrman, Frankowska, Guatteri, Hu, Li, Lü, Meng, Tang, Tessitore, Zhang, ...
  Major drawback in previous works: Strong assumptions on coefficients l and h, in particular excluding quadratic costs.

Spike Variation

Let $\bar{u}$ be optimal. Fix $\tau \in [0, T]$, $\varepsilon > 0$, $v \in U$, and set

$$u_t^\varepsilon := \begin{cases} v, & \tau \leq t \leq \tau + \varepsilon, \\ \bar{u}_t & \text{otherwise.} \end{cases}$$

Then

$$0 \leq J(u^\varepsilon) - J(\bar{u}) = \mathbb{E}\left[\int_0^T \int_\Lambda l(x_t^\varepsilon, u_t^\varepsilon) - l(\bar{x}_t, \bar{u}_t)\mathrm{d}\lambda\mathrm{d}t + \int_\Lambda h(x_T^\varepsilon) - h(\bar{x}_T)\mathrm{d}\lambda\right].$$

Roadmap: 1) Taylor expand integrands;
2) Divide by ε, send $\varepsilon \to 0$;
3) Identify remaining terms.

Because of stochastic calculus, we have to Taylor expand up to second order.

Quadratic Terms

Taylor expanding the cost functional to second order leads to the quadratic terms

$$\mathbb{E}\left[\int_0^T \int_\Lambda l_{xx}(\bar{x}_t(\lambda), \bar{u}_t) y_t^\varepsilon(\lambda) y_t^\varepsilon(\lambda) \mathrm{d}\lambda \mathrm{d}t + \int_\Lambda h_{xx}(\bar{x}_T(\lambda)) y_T^\varepsilon(\lambda) y_T^\varepsilon(\lambda) \mathrm{d}\lambda\right].$$

Idea: Linearize using tensor product. In finite dimensions, Peng derived equation for $y_t^\varepsilon \otimes y_t^\varepsilon$ on

$$\mathbb{R}^n \otimes \mathbb{R}^n \cong \mathbb{R}^{n\times n}.$$

In existing literature, this is generalized in infinite dimensions to

$$H \otimes H \cong L_2(H).$$

Problem: In order to perform duality analysis, we need to solve equation in $L_1(H)$.

Explicit Tensor Product

Instead, we use explicit representation

$$L^2(\Lambda) \otimes L^2(\Lambda) \cong L^2(\Lambda^2),$$

where a simple tensor $y \otimes z$ is identified with the function $(\lambda, \mu) \mapsto y(\lambda)z(\mu)$. Thus, we can rewrite quadratic terms as

$$\begin{aligned} &\mathbb{E}\left[\int_0^T \int_\Lambda l_{xx}(\bar{x}_t(\lambda), \bar{u}_t) y_t^\varepsilon(\lambda) y_t^\varepsilon(\lambda) \mathrm{d}\lambda \mathrm{d}t\right] \\ =&\mathbb{E}\left[\int_0^T \int_\Lambda l_{xx}(\bar{x}_t(\lambda), \bar{u}_t) \delta(y_t^\varepsilon \otimes y_t^\varepsilon)(\lambda) \mathrm{d}\lambda \mathrm{d}t\right], \end{aligned}$$

where $\delta : H_0^1(\Lambda^2) \to L^2(\Lambda)$ is defined by $\delta(w)(\lambda) := w(\lambda, \lambda)$. This expression is linear in $y_t^\varepsilon \otimes y_t^\varepsilon$.

Second Order Adjoint Equation

Theorem

The equation

$$\begin{cases} \mathrm{d}P_t(\lambda,\mu) = -[\Delta P_t(\lambda,\mu) + (b_x(\bar{x}_t(\lambda),\bar{u}_t) + b_x(\bar{x}_t(\mu),\bar{u}_t))P_t(\lambda,\mu) \\ \qquad + \langle \sigma_x(\bar{x}_t(\lambda),\bar{u}_t), \sigma_x(\bar{x}_t(\mu),\bar{u}_t)\rangle_{L_2(\Xi,\mathbb{R})} P_t(\lambda,\mu) \\ \qquad + \langle \sigma_x(\bar{x}_t(\lambda),\bar{u}_t) + \sigma_x(\bar{x}_t(\mu),\bar{u}_t), Q_t(\lambda,\mu)\rangle_{L_2(\Xi,\mathbb{R})} \\ \qquad + \delta^*(l_{xx}(\bar{x}_t(\lambda),\bar{u}_t)) + \delta^*(b_{xx}(\bar{x}_t(\lambda),\bar{u}_t)p_t(\lambda)) \\ \qquad + \delta^*(\langle \sigma_{xx}(\bar{x}_t(\lambda),\bar{u}_t), q_t\rangle_{L_2(\Xi,\mathbb{R})})]\mathrm{d}t + Q_t(\lambda,\mu)\mathrm{d}W_t \\ P_T(\lambda,\mu) = \delta^*(h_{xx}(\bar{x}_T(\lambda))) \end{cases}$$

has a unique adapted solution (P, Q)*, where*

$$P \in L^2([0,T]\times\Omega; L^2(\Lambda^2)) \cap L^2(\Omega; C([0,T]; H^{-1}(\Lambda^2))),$$

and

$$Q \in L^2([0,T]\times\Omega; L_2(\Xi; H^{-1}(\Lambda^2))).$$

Main Result

Let $(\bar{x}, \bar{u})$ *be an optimal pair. Then there exist adapted processes*

$$(p, q) \in L^2([0, T] \times \Omega; H_0^1(\Lambda)) \times L^2([0, T] \times \Omega; L_2(\Xi, L^2(\Lambda)))$$

satisfying the first order adjoint equation and adapted processes

$$(P, Q) \in L^2([0, T] \times \Omega; L^2(\Lambda^2)) \times L^2([0, T] \times \Omega; L_2(\Xi, H^{-1}(\Lambda^2)))$$

satisfying the second order adjoint equation such that

$$\inf_{u \in U} \mathcal{G}(t, \bar{x}_t, u) = \mathcal{G}(t, \bar{x}_t, \bar{u}_t),$$

for almost all $(t, \omega) \in [0, T] \times \Omega$*, where* $\mathcal{G} : [0, T] \times L^2(\Lambda) \times U \to \mathbb{R}$

$$\mathcal{G}(t, x, u) := \int_\Lambda l(x(\lambda), u) \mathrm{d}\lambda + \langle p_t, b(x, u) \rangle_{L^2(\Lambda)} + \frac{1}{2} tr(\sigma(x, u)^* P_t \sigma(x, u)) + tr\left(\sigma(x, u)^* \left[q_t - P_t \sigma(\bar{x}_t, \bar{u}_t)\right]\right).$$

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Viscosity Differentials of the Value Function in Optimal Control of Semilinear Stochastic Partial Differential Equations
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