Peng's Maximum Principle for Stochastic Partial Differential Equations

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Example

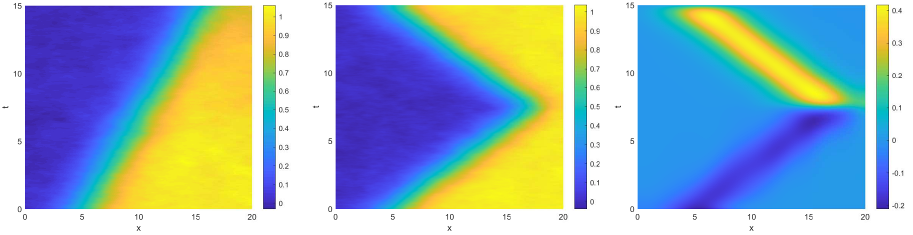
Consider stochastic Nagumo equation

$$\mathrm{d}x_t^u = [\Delta x_t^u + x_t^u(x_t^u - 1)(a - x_t^u) + u_t]\,\mathrm{d}t + \sigma \mathrm{d}W_t, \quad x_0^u = x \in L^2(\Lambda).$$

Introduce control $u : [0, T] \times \Lambda \to \mathbb{R}$. Minimize

$$J(u) = \mathbb{E}\left[\int_0^T \int_\Lambda \left(x_t^u(\lambda) - x_{\bar{\Lambda}}(t, \lambda) \right)^2 \mathrm{d}\lambda \mathrm{d}t + \int_\Lambda \left(x_T^u(\lambda) - x^T(\lambda) \right)^2 \mathrm{d}\lambda \right]$$

for given reference profiles $x_{\bar{\Lambda}}, x^T$.



Uncontrolled Solution | Controlled Solution | Optimal Control

Setting

Minimize

$$J(u) := \mathbb{E}\left[\int_0^T \int_\Lambda l(x_t^u(\lambda), u_t)\mathrm{d}\lambda\mathrm{d}t + \int_\Lambda h(x_T^u(\lambda))\mathrm{d}\lambda\right]$$

subject to

$$\begin{cases}\mathrm{d}x_t^u = [\Delta x_t^u + b(x_t^u, u_t)]\mathrm{d}t + \sigma(x_t^u, u_t)\mathrm{d}W_t, \quad t \in [0, T] \\ x_0^u = x \in L^2(\Lambda),\end{cases}$$

where

- l, h Nemytskii operators of (at most) quadratic growth;
- b, σ Nemytskii operators, Lipschitz;
- $(W_t)_{t\geq 0}$ cylindrical Wiener process;
- $\Lambda \subset \mathbb{R}$ bounded interval.

Goal: Derive necessary optimality condition.

Setting

Minimize

$$J(u) := \mathbb{E}\left[\int_0^T \int_\Lambda l(x_t^u(\lambda), u_t)\mathrm{d}\lambda\mathrm{d}t + \int_\Lambda h(x_T^u(\lambda))\mathrm{d}\lambda\right]$$

subject to

$$\begin{cases} \mathrm{d}x_t^u = [\Delta x_t^u + b(x_t^u, u_t)]\mathrm{d}t + \sigma(x_t^u, u_t)\mathrm{d}W_t, \quad t \in [0, T] \\ x_0^u = x \in L^2(\Lambda). \end{cases}$$

- Pontryagin's maximum principle (1956): Necessary optimality condition for controlled ODEs.
- Peng's maximum principle (1990): Generalization to controlled SDEs.
- Since then: Generalizations to controlled SPDEs by Du, Fuhrman, Frankowska, Guatteri, Hu, Li, Lü, Meng, Tang, Tessitore, Zhang, ... Previous results require strong assumptions on coefficients l and h excluding quadratic costs.

Spike Variation

Let $\bar{u}$ be optimal. Fix $\tau \in [0, T)$, $\varepsilon > 0$, $v \in U$, and set

$$u_t^\varepsilon := \begin{cases} v, & \tau \leq t \leq \tau + \varepsilon, \\ \bar{u}_t & \text{otherwise.} \end{cases}$$

Then

$$0 \leq J(u^\varepsilon) - J(\bar{u}) = \mathbb{E}\left[\int_0^T \int_\Lambda l(x_t^\varepsilon, u_t^\varepsilon) - l(\bar{x}_t, \bar{u}_t)\mathrm{d}\lambda\mathrm{d}t + \int_\Lambda h(x_T^\varepsilon) - h(\bar{x}_T)\mathrm{d}\lambda\right].$$

Roadmap: 1) Taylor expand integrands;
2) Divide by ε, send $\varepsilon \to 0$;
3) Identify remaining terms.

Because of stochastic calculus, we have to Taylor expand up to second order.

Quadratic Terms

Taylor expanding the cost functional to second order leads to the quadratic terms

$$\mathbb{E}\left[\int_0^T \int_\Lambda l_{xx}(\bar{x}_t(\lambda), \bar{u}_t) y_t^\varepsilon(\lambda) y_t^\varepsilon(\lambda) \mathrm{d}\lambda \mathrm{d}t + \int_\Lambda h_{xx}(\bar{x}_T(\lambda)) y_T^\varepsilon(\lambda) y_T^\varepsilon(\lambda) \mathrm{d}\lambda\right].$$

Peng's idea: Linearize using tensor product. Infinite-dimensional analogue:

$$H \otimes H \cong L_2(H).$$

Problem: Quadratic costs require duality analysis between $L_1(H)$ and $L(H)$.

Instead, we use explicit representation

$$L^2(\Lambda) \otimes L^2(\Lambda) \cong L^2(\Lambda^2).$$

Second Order Adjoint Equation

Theorem (Stannat, W., SICON 2021)

The equation

$$\begin{cases} \mathrm{d}P_t(\lambda,\mu) = -[\Delta P_t(\lambda,\mu) + (b_x(\bar{x}_t(\lambda),\bar{u}_t) + b_x(\bar{x}_t(\mu),\bar{u}_t))P_t(\lambda,\mu) \\ \qquad +\langle \sigma_x(\bar{x}_t(\lambda),\bar{u}_t), \sigma_x(\bar{x}_t(\mu),\bar{u}_t)\rangle_{L_2(\Xi,\mathbb{R})} P_t(\lambda,\mu) \\ \qquad +\langle \sigma_x(\bar{x}_t(\lambda),\bar{u}_t) + \sigma_x(\bar{x}_t(\mu),\bar{u}_t), Q_t(\lambda,\mu)\rangle_{L_2(\Xi,\mathbb{R})} \\ \qquad +\delta^*(l_{xx}(\bar{x}_t(\lambda),\bar{u}_t)) + \delta^*(b_{xx}(\bar{x}_t(\lambda),\bar{u}_t)p_t(\lambda)) \\ \qquad +\delta^*(\langle \sigma_{xx}(\bar{x}_t(\lambda),\bar{u}_t), q_t\rangle_{L_2(\Xi,\mathbb{R})})]\mathrm{d}t + Q_t(\lambda,\mu)\mathrm{d}W_t \\ P_T(\lambda,\mu) = \delta^*(h_{xx}(\bar{x}_T(\lambda))) \end{cases}$$

has a unique adapted solution (P,Q)*, where*

$$P \in L^2([0,T]\times\Omega; L^2(\Lambda^2)) \cap L^2(\Omega; C([0,T]; H^{-1}(\Lambda^2))),$$

and

$$Q \in L^2([0,T]\times\Omega; L_2(\Xi; H^{-1}(\Lambda^2))).$$

Peng's Maximum Principle for SPDEs

Theorem (Stannat, W., SICON 2021)

Let $(\bar{x}, \bar{u})$ be an optimal pair. Then there exist adapted processes

$$(p, q) \in L^2([0, T] \times \Omega; H_0^1(\Lambda)) \times L^2([0, T] \times \Omega; L_2(\Xi, L^2(\Lambda)))$$

satisfying the first order adjoint equation and adapted processes

$$(P, Q) \in L^2([0, T] \times \Omega; L^2(\Lambda^2)) \times L^2([0, T] \times \Omega; L_2(\Xi, H^{-1}(\Lambda^2)))$$

satisfying the second order adjoint equation such that

$$\inf_{u \in U} \mathcal{G}(t, \bar{x}_t, u) = \mathcal{G}(t, \bar{x}_t, \bar{u}_t),$$

for almost all $(t, \omega) \in [0, T] \times \Omega$, where $\mathcal{G} : [0, T] \times L^2(\Lambda) \times U \to \mathbb{R}$

$$\mathcal{G}(t, x, u) := \int_\Lambda l(x(\lambda), u) \mathrm{d}\lambda + \langle p_t, b(x, u) \rangle_{L^2(\Lambda)} + tr(\sigma(x, u)^* q_t) + \frac{1}{2} tr(\sigma(x, u)^* P_t \sigma(x, u)) - tr(\sigma(x, u)^* P_t \sigma(\bar{x}_t, \bar{u}_t)).$$

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