# Necessary and Sufficient Conditions for Optimal Control of Semilinear SPDEs

Lukas Wessels

joint work with Wilhelm Stannat

*www.lukaswessels.org*

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Georgia Tech

SFB 910

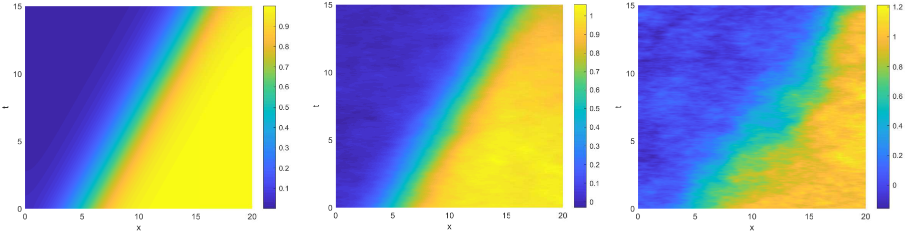
Technische Universität Berlin

# Example

Consider the Nagumo equation

$$\mathrm{d}x_t = [\Delta x_t + x_t(x_t - 1)(a - x_t)]\,\mathrm{d}t + \sigma \mathrm{d}W_t, \quad x_0 = x \in L^2(\Lambda),$$

with Neumann boundary conditions, where $\Lambda \subset \mathbb{R}$ is bounded and $a \in (0,1)$.




$\sigma = 0$ $\sigma = \frac{1}{2}$ $\sigma = 2$

# Control of the Stochastic Nagumo Equation

Fix $T > 0$. Introduce control $u : [0, T] \times \Lambda \times \Omega \to \mathbb{R}$

$$\begin{cases} \mathrm{d}x_t^u = [\Delta x_t^u + x_t^u(x_t^u - 1)(a - x_t^u) + u_t]\,\mathrm{d}t + \sigma \mathrm{d}W_t \\ x_0^u = x \in L^2(\Lambda), \end{cases} \qquad (\star)$$

and cost functional

$$J(u) = \mathbb{E}\left[\int_0^T \int_\Lambda \frac{c_{\overline{\Lambda}}}{2}\left(x_t^u(\lambda) - x_{\overline{\Lambda}}(t,\lambda)\right)^2 + \frac{\nu}{2} u^2(t,\lambda)\mathrm{d}\lambda\mathrm{d}t + \int_\Lambda \frac{c_T}{2}\left(x_T^u(\lambda) - x^T(\lambda)\right)^2 \mathrm{d}\lambda\right]$$

where

- $x_{\overline{\Lambda}}$, $x^T$ – desired reference profiles
- $c_{\overline{\Lambda}}, \nu, c_T \geq 0$.

Goal:

$$\boxed{\text{Minimize} \quad J \quad \text{subject to } (\star)}$$

# Setting

Minimize

$$J(u) := \mathbb{E}\left[\int_0^T \int_\Lambda l(x_t^u(\lambda), u_t)\mathrm{d}\lambda\mathrm{d}t + \int_\Lambda h(x_T^u(\lambda))\mathrm{d}\lambda\right]$$

subject to

$$\begin{cases} \mathrm{d}x_t^u = [\Delta x_t^u + b(x_t^u, u_t)]\mathrm{d}t + \sigma(x_t^u, u_t)\mathrm{d}W_t, \quad t \in [0, T] \\ x_0^u = x_0 \in L^2(\Lambda), \end{cases}$$

where

- $l, h$ Nemytskii operators of (at most) quadratic growth
- $b, \sigma$ Nemytskii operators, Lipschitz
- $(W_t)_{t\geq 0}$ cylindrical Wiener process
- $\Lambda \subset \mathbb{R}^d$ bounded domain
- control domain $U$ non-convex.

# Outline

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# Pontryagin 1956: Controlled ODEs

Minimize

$$J(u) := \int_0^T l(x_t^u, u_t)\mathrm{d}t + h(x_T^u)$$

subject to

$$\begin{cases} \partial_t x_t^u = b(x_t^u, u_t), & t \in [0, T] \\ x_0^u = x_0 \in \mathbb{R}^n. \end{cases}$$

Let $\bar{u}$ be optimal. Then

$$\inf_{u \in U} H(t, \bar{x}_t, u) = H(t, \bar{x}_t, \bar{u}_t),$$

for almost every $t \in [0, T]$, where Hamiltonian $H : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$

$$H(t, x, u) := l(x, u) + \langle p_t, b(x, u) \rangle$$

where $p_t$ is the adjoint state.

# Peng 1990: Controlled SDEs

Minimize

$$J(u) := \mathbb{E}\left[\int_0^T l(x_t^u, u_t)\mathrm{d}t + h(x_T^u)\right]$$

subject to

$$\begin{cases} \mathrm{d}x_t^u = b(x_t^u, u_t)\mathrm{d}t + \sigma(x_t^u, u_t)\mathrm{d}W_t, & t \in [0, T] \\ x_0^u = x_0 \in \mathbb{R}^n. \end{cases}$$

Let $\bar{u}$ be optimal. Then

$$\inf_{u \in U} \mathcal{G}(t, \bar{x}_t, u) = \mathcal{G}(t, \bar{x}_t, \bar{u}_t),$$

for almost all $(t, \omega) \in [0, T] \times \Omega$, where generalized Hamiltonian $\mathcal{G} : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$

$$\mathcal{G}(t, x, u) := H(t, x, u) + \frac{1}{2}\mathrm{tr}(\sigma(x, u)^* P_t \sigma(x, u)) + \mathrm{tr}\left(\sigma(x, u)^* \left[q_t - P_t \sigma(\bar{x}_t, \bar{u}_t)\right]\right)$$

where $(p_t, q_t)$ first order adjoint state and $P_t \in \mathbb{R}^{n \times n}$ second order adjoint state.

# Now: Controlled SPDEs

Minimize

$$J(u) := \mathbb{E}\left[\int_0^T \int_\Lambda l(x_t^u(\lambda), u_t)\mathrm{d}\lambda\mathrm{d}t + \int_\Lambda h(x_T^u(\lambda))\mathrm{d}\lambda\right]$$

subject to

$$\begin{cases} \mathrm{d}x_t^u = [\Delta x_t^u + b(x_t^u, u_t)]\mathrm{d}t + \sigma(x_t^u, u_t)\mathrm{d}W_t, & t \in [0, T] \\ x_0^u = x_0 \in L^2(\Lambda). \end{cases}$$

- Since Peng's result: Many generalizations to infinite-dimensional stochastic case by Du, Fuhrman, Frankowska, Guatteri, Hu, Li, Lü, Meng, Tang, Tessitore, Zhang, ...
- Major drawback in previous works: Restrictive assumptions on coefficients $l$ and $h$, in particular excluding quadratic costs.
- Main obstacle: Characterization of second order adjoinst state $P_t \in L(H)$.

# Spike Variation

Let $\bar{u}$ be optimal. Fix $\tau \in [0, T]$, $\varepsilon > 0$, $v \in U$, and set

$$u_t^\varepsilon := \begin{cases} v, & \tau \leq t \leq \tau + \varepsilon, \\ \bar{u}_t & \text{otherwise.} \end{cases}$$

Then

$$0 \leq J(u^\varepsilon) - J(\bar{u}) = \mathbb{E}\left[\int_0^T \int_\Lambda l(x_t^\varepsilon, u_t^\varepsilon) - l(\bar{x}_t, \bar{u}_t) \mathrm{d}\lambda \mathrm{d}t + \int_\Lambda h(x_T^\varepsilon) - h(\bar{x}_T) \mathrm{d}\lambda\right].$$

Taylor expansions: For terminal costs we have

$$\mathbb{E}\left[\int_\Lambda h(x_T^\varepsilon) - h(\bar{x}_T)\mathrm{d}\lambda\right] \approx \mathbb{E}\left[\int_\Lambda h_x(\bar{x}_T) y_T^\varepsilon \mathrm{d}\lambda\right],$$

where $y^\varepsilon$ satisfies linearized state equation

$$\begin{cases} \mathrm{d}y_t^\varepsilon = \left[\Delta y_t^\varepsilon + b_x(\bar{x}_t, \bar{u}_t) y_t^\varepsilon + b(\bar{x}_t, u_t^\varepsilon) - b(\bar{x}_t, \bar{u}_t)\right] \mathrm{d}t \\ \qquad\qquad + \left[\sigma_x(\bar{x}_t, \bar{u}_t) y_t^\varepsilon + \sigma(\bar{x}_t, u_t^\varepsilon) - \sigma(\bar{x}_t, \bar{u}_t)\right] \mathrm{d}W_t \\ y_0^\varepsilon = 0. \end{cases}$$

# Asymptotic of First Order Taylor Expansions

Applying Itô's formula to $\|y_t^\varepsilon\|^2_{L^2(\Lambda)}$ yields

$$\begin{aligned}\|y_t^\varepsilon\|^2_{L^2(\Lambda)} =& 2\int_0^t \langle \Delta y_s^\varepsilon + b_x(\bar{x}_s,\bar{u}_s)y_s^\varepsilon + b(\bar{x}_s,u_s^\varepsilon) - b(\bar{x}_s,\bar{u}_s), y_s^\varepsilon\rangle \mathrm{d}s \\ &+ 2\int_0^t \langle y_s^\varepsilon, (\sigma_x(\bar{x}_s,\bar{u}_s)y_s^\varepsilon + \sigma(\bar{x}_s,u_s^\varepsilon) - \sigma(\bar{x}_s,\bar{u}_s))\mathrm{d}W_s\rangle \\ &+ \int_0^t \|\sigma_x(\bar{x}_s,\bar{u}_s)y_s^\varepsilon + \sigma(\bar{x}_s,u_s^\varepsilon) - \sigma(\bar{x}_s,\bar{u}_s)\|^2_{L_2(\Xi,L^2(\Lambda))}\mathrm{d}s.\end{aligned}$$

Itô correction term

$$\begin{aligned}&\int_0^t \|\sigma_x(\bar{x}_s,\bar{u}_s)y_s^\varepsilon + \sigma(\bar{x}_s,u_s^\varepsilon) - \sigma(\bar{x}_s,\bar{u}_s)\|^2_{L_2(\Xi,L^2(\Lambda))}\mathrm{d}s \\ &\leq 2\int_0^t \|\sigma_x(\bar{x}_s,\bar{u}_s)y_s^\varepsilon\|^2_{L_2(\Xi,L^2(\Lambda))}\mathrm{d}s + 2\int_\tau^{\tau+\varepsilon} \|\sigma(\bar{x}_s,v) - \sigma(\bar{x}_s,\bar{u}_s)\|^2_{L_2(\Xi,L^2(\Lambda))}\mathrm{d}s.\end{aligned}$$

The last term is of order $\mathcal{O}(\varepsilon)$; we need $o(\varepsilon)$. $\rightsquigarrow$ need second order Taylor expansions!

# Variational Equations

First order:

$$\begin{cases} \mathrm{d}y_t^\varepsilon = [\Delta y_t^\varepsilon + b_x(\bar{x}_t, \bar{u}_t)y_t^\varepsilon + b(\bar{x}_t, u_t^\varepsilon) - b(\bar{x}_t, \bar{u}_t)]\,\mathrm{d}t \\ \qquad\qquad + [\sigma_x(\bar{x}_t, \bar{u}_t)y_t^\varepsilon + \sigma(\bar{x}_t, u_t^\varepsilon) - \sigma(\bar{x}_t, \bar{u}_t)]\,\mathrm{d}W_t \\ y_0^\varepsilon = 0. \end{cases}$$

Second order:

$$\begin{cases} \mathrm{d}z_t^\varepsilon = \left[\Delta z_t^\varepsilon + b_x(\bar{x}_t, \bar{u}_t)z_t^\varepsilon + \frac{1}{2}b_{xx}(\bar{x}_t, \bar{u}_t)y_t^\varepsilon y_t^\varepsilon + (b_x(\bar{x}_t, u_t^\varepsilon) - b_x(\bar{x}_t, \bar{u}_t))y_t^\varepsilon\right]\mathrm{d}t \\ \qquad\qquad + \left[\sigma_x(\bar{x}_t, \bar{u}_t)z_t^\varepsilon + \frac{1}{2}\sigma_{xx}(\bar{x}_t, \bar{u}_t)y_t^\varepsilon y_t^\varepsilon + (\sigma_x(\bar{x}_t, u_t^\varepsilon) - \sigma_x(\bar{x}_t, \bar{u}_t))y_t^\varepsilon\right]\mathrm{d}W_t \\ z_0^\varepsilon = 0. \end{cases}$$

## Lemma

*It holds that*

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|y_t^\varepsilon\|_{L^2(\Lambda)}^2\right] \leq C\varepsilon \qquad \textit{and} \qquad \sup_{t\in[0,T]} \mathbb{E}\left[\|z_t^\varepsilon\|_{L^2(\Lambda)}^2\right] \leq C\varepsilon^2.$$

# Second Order Expansion of Cost Functional

From

$$0 \leq J(u^\varepsilon) - J(\bar{u}) = \mathbb{E}\left[\int_0^T \int_\Lambda l(x_t^\varepsilon, u_t^\varepsilon) - l(\bar{x}_t, \bar{u}_t)\mathrm{d}\lambda\mathrm{d}t + \int_\Lambda h(x_T^\varepsilon) - h(\bar{x}_T)\mathrm{d}\lambda\right]$$

we derive:

## Lemma

*It holds*

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \int_\Lambda l_x(\bar{x}_t(\lambda), \bar{u}_t)(y_t^\varepsilon(\lambda) + z_t^\varepsilon(\lambda)) + \frac{1}{2} l_{xx}(\bar{x}_t(\lambda), \bar{u}_t) y_t^\varepsilon(\lambda) y_t^\varepsilon(\lambda)\mathrm{d}\lambda\mathrm{d}t\right] \\
+&\mathbb{E}\left[\int_\Lambda h_x(\bar{x}_T(\lambda))(y_T^\varepsilon(\lambda) + z_T^\varepsilon(\lambda)) + \frac{1}{2} h_{xx}(\bar{x}_T(\lambda)) y_T^\varepsilon(\lambda) y_T^\varepsilon(\lambda)\mathrm{d}\lambda\right] \\
+&\mathbb{E}\left[\int_0^T \int_\Lambda l(\bar{x}_t(\lambda), u_t^\varepsilon) - l(\bar{x}_t(\lambda), \bar{u}_t)\mathrm{d}\lambda\mathrm{d}t\right] \geq o(\varepsilon).
\end{aligned}$$

Next step: Separate dependence on $\varepsilon$.

# First Order Adjoint State

Consider SPDE

$$\begin{cases} \mathrm{d}y_t = [\Delta y_t + b_x(\bar{x}_t, \bar{u}_t)y_t + \varphi_t]\,\mathrm{d}t + [\sigma_x(\bar{x}_t, \bar{u}_t)y_t + \psi_t]\,\mathrm{d}W_t \\ y_0 = 0, \end{cases}$$

where $(\varphi, \psi) \in L^2([0,T] \times \Omega; L^2(\Lambda)) \times L^2([0,T] \times \Omega; L_2(\Xi, L^2(\Lambda)))$.
Construct linear functional

$$\mathcal{T}(\varphi, \psi) := \mathbb{E}\left[\int_0^T \int_\Lambda l_x(\bar{x}_t(\lambda), \bar{u}_t)y_t(\lambda)\mathrm{d}\lambda\mathrm{d}t + \int_\Lambda h_x(\bar{x}_T(\lambda))y_T(\lambda)\mathrm{d}\lambda\right].$$

By Riesz's representation theorem, there is a unique pair

$$(p, q) \in L^2([0,T] \times \Omega; L^2(\Lambda)) \times L^2([0,T] \times \Omega; L_2(\Xi, L^2(\Lambda)))$$

such that

$$\mathcal{T}(\varphi, \psi) = \mathbb{E}\left[\int_0^T \langle \varphi_t, p_t\rangle_{L^2(\Lambda)} + \langle \psi_t, q_t\rangle_{L_2(\Xi, L^2(\Lambda))}\mathrm{d}t\right]$$

for all $(\varphi, \psi) \in L^2([0,T] \times \Omega; L^2(\Lambda)) \times L^2([0,T] \times \Omega; L_2(\Xi, L^2(\Lambda)))$.

# First Order Adjoint Equation

Adjoint state property:

$$\mathbb{E}\left[\int_0^T \langle y_t, l_x(\bar{x}_t, \bar{u}_t)\rangle \mathrm{d}t + \langle h_x(\bar{x}_T), y_T\rangle\right] = \mathbb{E}\left[\int_0^T \langle p_t, \varphi_t\rangle + \langle q_t, \psi_t\rangle \mathrm{d}t\right].$$

Applying Itô's product rule yields

$$\begin{aligned}\mathrm{d}\langle p_t, y_t\rangle_{L^2(\Lambda)} =& \langle p_t, \mathrm{d}y_t\rangle_{L^2(\Lambda)} + \langle y_t, \mathrm{d}p_t\rangle_{L^2(\Lambda)} + \mathrm{d}\langle p, y\rangle_t \\ \overset{!}{=}& \left[\langle p_t, \varphi_t\rangle_{L^2(\Lambda)} + \langle q_t, \psi_t\rangle_{L_2(\Xi, L^2(\Lambda))} - \langle y_t, l_x(\bar{x}_t, \bar{u}_t)\rangle_{L^2(\Lambda)}\right] \mathrm{d}t + dM_t.\end{aligned}$$

for some martingale $(M_t)_{t\geq 0}$. Thus,

$$\begin{cases}\mathrm{d}p_t = -\left[\Delta p_t + b_x(\bar{x}_t, \bar{u}_t)p_t + l_x(\bar{x}_t, \bar{u}_t) + \langle \sigma_x(\bar{x}_t, \bar{u}_t), q_t\rangle_{L_2(\Xi, \mathbb{R})}\right] \mathrm{d}t + q_t \mathrm{d}W_t \\ p_T = h_x(\bar{x}_T).\end{cases}$$

Unique variational solution $(p, q)$, where

$$p \in L^2([0, T] \times \Omega; H_0^1(\Lambda)) \cap L^2(\Omega; C([0, T]; L^2(\Lambda)))$$

and

$$q \in L^2([0, T] \times \Omega; L_2(\Xi, L^2(\Lambda))).$$

# Quadratic Terms

Follow same route for quadratic terms

$$\mathbb{E}\left[\int_0^T \int_\Lambda l_{xx}(\bar{x}_t(\lambda), \bar{u}_t) y_t^\varepsilon(\lambda) y_t^\varepsilon(\lambda) \mathrm{d}\lambda \mathrm{d}t + \int_\Lambda h_{xx}(\bar{x}_T(\lambda)) y_T^\varepsilon(\lambda) y_T^\varepsilon(\lambda) \mathrm{d}\lambda\right].$$

Peng's idea in finite dimensions: Linearize using tensor product $y_t^\varepsilon \otimes y_t^\varepsilon$ and derive equation on

$$\mathbb{R}^n \otimes \mathbb{R}^n \cong \mathbb{R}^{n\times n}.$$

Infinite dimensional analogue:

$$H \otimes H \cong L_2(H).$$

Problem: Quadratic costs require duality analysis between $L_1(H)$ and $L(H)$.

# Explicit Tensor Product

Instead, we use explicit representation

$$
\begin{aligned}
L^2(\Lambda) \otimes L^2(\Lambda) &\cong L^2(\Lambda^2) \\
y \otimes z &\leftrightsquigarrow ((\lambda, \mu) \mapsto y(\lambda) z(\mu)) .
\end{aligned}
$$

Rewrite quadratic terms as

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T \int_\Lambda l_{xx}(\bar{x}_t(\lambda), \bar{u}_t) y_t^\varepsilon(\lambda) y_t^\varepsilon(\lambda) \mathrm{d}\lambda \mathrm{d}t\right] \\
&= \mathbb{E}\left[\int_0^T \int_\Lambda l_{xx}(\bar{x}_t(\lambda), \bar{u}_t) \delta(y_t^\varepsilon \otimes y_t^\varepsilon)(\lambda) \mathrm{d}\lambda \mathrm{d}t\right],
\end{aligned}
$$

where

$$
\begin{aligned}
\delta : H_0^1(\Lambda^2) &\to L^2(\Lambda) \\
w &\mapsto (\lambda \mapsto w(\lambda, \lambda)).
\end{aligned}
$$

# Second Order Adjoint Equation

## Theorem (Stannat, W., SICON 2021)

*The equation*

$$\begin{cases} \mathrm{d}P_t(\lambda,\mu) = -[\Delta P_t(\lambda,\mu) + (b_x(\bar{x}_t(\lambda),\bar{u}_t) + b_x(\bar{x}_t(\mu),\bar{u}_t))P_t(\lambda,\mu) \\ \qquad +\langle \sigma_x(\bar{x}_t(\lambda),\bar{u}_t), \sigma_x(\bar{x}_t(\mu),\bar{u}_t)\rangle_{L_2(\Xi,\mathbb{R})} P_t(\lambda,\mu) \\ \qquad +\langle \sigma_x(\bar{x}_t(\lambda),\bar{u}_t) + \sigma_x(\bar{x}_t(\mu),\bar{u}_t), Q_t(\lambda,\mu)\rangle_{L_2(\Xi,\mathbb{R})} \\ \qquad +\delta^*(l_{xx}(\bar{x}_t(\lambda),\bar{u}_t)) + \delta^*(b_{xx}(\bar{x}_t(\lambda),\bar{u}_t)p_t(\lambda)) \\ \qquad +\delta^*(\langle \sigma_{xx}(\bar{x}_t(\lambda),\bar{u}_t), q_t\rangle_{L_2(\Xi,\mathbb{R})})]\mathrm{d}t + Q_t(\lambda,\mu)\mathrm{d}W_t \\ P_T(\lambda,\mu) = \delta^*(h_{xx}(\bar{x}_T(\lambda))) \end{cases}$$

*has a unique adapted solution* $(P,Q)$*, where*

$$P \in L^2([0,T]\times\Omega; L^2(\Lambda^2)) \cap L^2(\Omega; C([0,T]; H^{-1}(\Lambda^2))),$$

*and*

$$Q \in L^2([0,T]\times\Omega; L_2(\Xi; H^{-1}(\Lambda^2))).$$

# Peng's Maximum Principle for SPDEs

## Theorem (Stannat, W., SICON 2021)

*Let* $(\bar{x}, \bar{u})$ *be an optimal pair. Then there exist adapted processes*

$$(p, q) \in L^2([0, T] \times \Omega; H_0^1(\Lambda)) \times L^2([0, T] \times \Omega; L_2(\Xi, L^2(\Lambda)))$$

*satisfying the first order adjoint equation and adapted processes*

$$(P, Q) \in L^2([0, T] \times \Omega; L^2(\Lambda^2)) \times L^2([0, T] \times \Omega; L_2(\Xi, H^{-1}(\Lambda^2)))$$

*satisfying the second order adjoint equation such that*

$$\inf_{u \in U} \mathcal{G}(t, \bar{x}_t, u) = \mathcal{G}(t, \bar{x}_t, \bar{u}_t),$$

*for almost all* $(t, \omega) \in [0, T] \times \Omega$*, where* $\mathcal{G} : [0, T] \times L^2(\Lambda) \times U \to \mathbb{R}$

$$\mathcal{G}(t, x, u) := \int_\Lambda l(x(\lambda), u) \mathrm{d}\lambda + \langle p_t, b(x, u) \rangle_{L^2(\Lambda)} + tr(\sigma(x, u)^* q_t) + \frac{1}{2} tr(\sigma(x, u)^* P_t \sigma(x, u)) - tr(\sigma(x, u)^* P_t \sigma(\bar{x}_t, \bar{u}_t)).$$

# Dynamic Programming Approach

Minimize

$$J(s,x;u) := \mathbb{E}\left[\int_s^T \int_\Lambda l(x_t^u(\lambda), u_t)\mathrm{d}\lambda\mathrm{d}t + \int_\Lambda h(x_T^u(\lambda))\mathrm{d}\lambda\right]$$

over $u \in \mathcal{U}_s$ subject to

$$\begin{cases} \mathrm{d}x_t^u = [\Delta x_t^u + b(x_t^u, u_t)]\mathrm{d}t + \sigma(x_t^u, u_t)\mathrm{d}W_t, & t \in [s, T] \\ x_s^u = x \in L^2(\Lambda). \end{cases}$$

Introduce value function $V : [0, T] \times L^2(\Lambda) \to \mathbb{R}$,

$$V(s,x) := \inf_{u \in \mathcal{U}_s} J(s,x;u).$$

Satisfies dynamic programming principle

$$V(s,x) = \inf_{u \in \mathcal{U}_s} \mathbb{E}\left[\int_s^\tau l(x_t^u, u_t)\mathrm{d}t + V(\tau, x_\tau^u)\right], \qquad \forall \tau \in [s, T].$$

Can be used to derive optimality conditions.

# Link Between Value Function and Adjoint States

Under smoothness assumptions, it holds

$$\begin{cases} V_s(t,\bar{x}_t) = -\langle \Delta \bar{x}_t, DV(t,\bar{x}_t)\rangle_{L^2(\Lambda)} - \mathcal{H}(\bar{x}_t, \bar{u}_t, DV(t,\bar{x}_t), D^2V(t,\bar{x}_t)) \\ DV(t,\bar{x}_t) = p_t \\ D^2V(t,\bar{x}_t)\sigma(\bar{x}_t,\bar{u}_t) = q_t, \end{cases}$$

where

$$\mathcal{H}(x,u,p,P) := \int_\Lambda l(x(\lambda),u)\mathrm{d}\lambda + \langle p, b(x,u)\rangle_{L^2(\Lambda)} + \frac{1}{2}\mathrm{tr}\left(\sigma(x,u)^* P \sigma(x,u)\right).$$

Generalizations dropping smoothness assumptions and using viscosity differentials up to first order in infinite dimensions by Cannarsa, Frankowska, Zhou, ...

# Viscosity Differentials

If $V \in C^{1,2}([0,T] \times L^2(\Lambda))$, it holds

$$\lim_{\tau \downarrow t, z \to x} \frac{1}{|\tau - t| + \|z - x\|^2} \Big[ V(\tau, z) - V(t,x) - \partial_t V(t,x)(\tau - t) - \langle DV(t,x), z - x \rangle_{L^2(\Lambda)} - \frac{1}{2} \langle z - x, D^2 V(t,x)(z-x) \rangle_{L^2(\Lambda)} \Big] = 0.$$

Weaker notion of differentiability:

## Definition (Viscosity Superdifferential)

*We say* $(G, p, P) \in D^{1,2,+}_{t+,x} V(t,x)$ *if*

$$\limsup_{\tau \downarrow t, z \to x} \frac{1}{|\tau - t| + \|z - x\|^2} \Big[ V(\tau, z) - V(t,x) - G(\tau - t) - \langle p, z - x \rangle_{L^2(\Lambda)} - \frac{1}{2} \langle z - x, P(z-x) \rangle_{L^2(\Lambda)} \Big] \leq 0.$$

# Link Between Value Function and Adjoint States

## Theorem (Stannat, W. (2022+))

*For almost every $t \in [0, T]$, it holds*

$$[-\langle \Delta \bar{x}_t, p_t \rangle_{H^{-1}(\Lambda) \times H_0^1(\Lambda)} - \mathcal{G}(t, \bar{x}_t, \bar{u}_t), \infty) \times \{p_t\} \times [P_t, \infty) \subset D^{1,2,+}_{t+,x} V(t, \bar{x}_t)$$

*$\mathbb{P}$-almost surely.*

## Corollary (Stannat, W. (2022+))

*It holds for almost all $t \in [s, T]$*

$$\mathcal{G}(t, \bar{x}_t, \bar{u}_t) \leq \mathcal{H}(t, \bar{x}_t, \bar{u}_t, p_t, P_t),$$

*$\mathbb{P}$-almost surely, i.e.,*

$$tr(\sigma(\bar{x}_t, \bar{u}_t)(q_t - P_t \sigma(\bar{x}_t, \bar{u}_t))) \leq 0.$$

# Viscosity Solutions

Value function formally satisfies Hamilton-Jacobi-Bellman equation

$$\begin{cases} V_s + \langle \Delta x, DV \rangle_{L^2(\Lambda)} + \inf_{u \in U} \mathcal{H}(x, u, DV, D^2 V) = 0, & (s, x) \in [0, T] \times L^2(\Lambda) \\ V(T, x) = \int_\Lambda h(x(\lambda)) \mathrm{d}\lambda, & x \in L^2(\Lambda) \end{cases}$$

where

$$\mathcal{H}(x, u, p, P) := \int_\Lambda l(x(\lambda), u) \mathrm{d}\lambda + \langle p, b(x, u) \rangle_{L^2(\Lambda)} + \frac{1}{2} \mathrm{tr} \left( \sigma(x, u)^* P \sigma(x, u) \right).$$

## Definition (Viscosity Solution, Bounded Case)

*$V$ is a viscosity subsolution, if*

- $V(T, x) \leq \int_\Lambda h(x(\lambda)) \mathrm{d}\lambda, \quad x \in L^2(\Lambda);$
- $\forall \phi \in C^{1,2}((s, T) \times L^2(\Lambda))$ *such that* $V - \phi$ *attains maximum at* $(t, x)$, *it holds*

$$\phi_s(t, x) + \langle A x, D\phi(t, x) \rangle + \inf_{u \in U} \mathcal{H}(x, u, D\phi(t, x), D^2 \phi(t, x)) \geq 0.$$

# Viscosity Solutions II

It holds:

$$\boxed{(G, p, P) \in D^{1,2,+}_{t,x} v(t, x)}$$

$$\Longleftrightarrow$$

$\exists \phi \in C^{1,2}((s, T) \times L^2(\Lambda))$ such that:

1. $v - \phi$ attains maximum at $(t, x)$,
2. $(\phi(t, x), \partial_t \phi(t, x), D\phi(t, x), D^2\phi(t, x)) = (v(t, x), G, p, P)$.

Equivalent definition of viscosity solution in the bounded case (!):

## Definition (Viscosity Solution, Bounded Case)

*$V$ is viscosity subsolution, if*

- $V(T, x) \leq \int_\Lambda h(x(\lambda))\mathrm{d}\lambda, \quad x \in L^2(\Lambda);$
- *for every* $(G, p, P) \in D^{1,2,+}_{t,x} V(t, x)$

$$G + \langle Ax, p\rangle + \inf_{u \in U} \mathcal{H}(x, u, p, P) \geq 0.$$

# Generalized Hamiltonian vs. Hamiltonian

## Corollary (Stannat, W. (2022+))

*It holds for almost all $t \in [s, T]$, $\mathbb{P}$-almost surely,*

$$\mathcal{G}(t, \bar{x}_t, \bar{u}_t) \leq \mathcal{H}(t, \bar{x}_t, \bar{u}_t, p_t, P_t).$$

Proof (bounded case!): Since $V$ is viscosity solution of HJB equation, we have for every $(G, p, P) \in D^{1,2,+}_{t,x} V(t, \bar{x}_t)$:

$$0 \leq G + \langle Ax, p \rangle + \inf_{u \in U} \mathcal{H}(x, u, p, P).$$

Thus,

$$\begin{aligned} 0 &\leq -\mathcal{G}(t, \bar{x}_t, \bar{u}_t) - \langle A\bar{x}_t, p_t \rangle + \langle A\bar{x}_t, p_t \rangle + \inf_{u \in U} \mathcal{H}(t, \bar{x}_t, u, p_t, P_t) \\ &\leq -\mathcal{G}(t, \bar{x}_t, \bar{u}_t) + \mathcal{H}(t, \bar{x}_t, \bar{u}_t, p_t, P_t). \end{aligned}$$

□

# Generalized Hamiltonian vs. Hamiltonian

HJB equation:

$$\begin{cases} V_s + \langle \Delta x, DV \rangle_{L^2(\Lambda)} + \inf_{u \in U} \mathcal{H}(x, u, DV, D^2 V) = 0, \quad (s, x) \in [0, T] \times L^2(\Lambda) \\ V(T, x) = \int_\Lambda h(x(\lambda)) \mathrm{d}\lambda, \quad x \in L^2(\Lambda) \end{cases}$$

Problematic term:

$$\langle \Delta x, D\phi(t, x) \rangle_{L^2(\Lambda)}, \qquad x \in L^2(\Lambda).$$

$\rightsquigarrow$ Need to restrict class of test functions (see $B$-continuous viscosity solutions).

We only need to make sense of

$$\langle \Delta \bar{x}_t, D\phi(t, \bar{x}_t) \rangle_{L^2(\Lambda)},$$

where $\bar{x}_t \in H_0^1(\Lambda)$, $\mathrm{d}t \otimes \mathbb{P}$-a.s.

# Verification Theorem

Minimize

$$J(s, x; u) := \mathbb{E}\left[\int_s^T L(x_t^u, u_t)\mathrm{d}t + H(x_T^u)\right]$$

over $u \in \mathcal{U}_s$ subject to

$$\begin{cases} \mathrm{d}x_t^u = [\Delta x_t^u + B(x_t^u, u_t)]\mathrm{d}t + \Sigma(x_t^u, u_t)\mathrm{d}W_t, \quad t \in [s, T] \\ x_s^u = x \in L^2(\Lambda). \end{cases}$$

Smooth verification theorem: Let $(x^*, u^*)$ be admissible such that

$$u_t^* \in \arg\min_{u \in U} \mathcal{H}(x_t^*, u, DV(t, x_t^*), D^2V(t, x_t^*)),$$

for almost every $(t, \omega)$. Then $(x^*, u^*)$ is optimal.

Verification theorems in the stochastic case in the framework of viscosity solutions by Fabbri, Gozzi, Li, Święch, Yong, Zhou, . . . .

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Smooth verification theorem: Let $(x^*, u^*)$ be admissible such that

$$V_s(t, x_t^*) + \langle \Delta x_t^*, DV(t, x_t^*)\rangle_{L^2(\Lambda)} + \mathcal{H}(x^*, u_t^*, DV(t, x_t^*), D^2V(t, x_t^*)) = 0$$

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# Verification Theorem

## Theorem (Stannat, W. (2022+))

*Assume*

- $\|\Sigma(x,u)\|_{L_2(\Xi, H_0^1(\Lambda))} \leq C(1 + \|x\|_{H_0^1(\Lambda)})$
- $V(t+\tau, x) - V(t,x) \leq C(1 + \|x\|^2_{H_0^1(\Lambda)})\tau$
- $V(t,\cdot) - C\|\cdot\|^2_{L^2(\Lambda)}$ *is concave.*

*Let* $(x^*, u^*)$ *be an admissible pair. Suppose there are adapted processes* $(G, p, P)$ *taking values in* $\mathbb{R}$, $H_0^1(\Lambda)$ *and* $L_2(L^2(\Lambda))$, *such that for almost all* $t \in [s, T]$:

$$(G_t, p_t, P_t) \in D^{1,2,+}_{t+,x} V(t, x_t^*)$$

$\mathbb{P}$*-almost surely, and*

$$\mathbb{E}\left[\int_s^T G_t + \langle \Delta x_t^*, p_t\rangle_{H^{-1}(\Lambda)\times H_0^1(\Lambda)} + \mathcal{H}(x_t^*, u_t^*, p_t, P_t)\mathrm{d}t\right] \geq 0.$$

*Then* $(x^*, u^*)$ *is an optimal pair.*

# Approximation of Optimal Controls

Numerical approximations by Dunst, Majee, Prohl, Vallet, . . . .

Minimize

$$J(u)=\mathbb{E}\left[\int_0^T\int_\Lambda \frac{c_{\overline{\Lambda}}}{2}\left(x_t^u(\lambda)-x_{\overline{\Lambda}}(t,\lambda)\right)^2+\frac{\nu}{2}u^2(t,\lambda)\mathrm{d}\lambda\mathrm{d}t+\int_\Lambda \frac{c_T}{2}\left(x_T^u(\lambda)-x^T(\lambda)\right)^2\mathrm{d}\lambda\right]$$

over $u \in L^2([0, T] \times \Lambda)$ subject to

$$\mathrm{d}x_t^u = [\Delta x_t^u + b(x_t^u) + u_t]\,\mathrm{d}t + \sigma \mathrm{d}W_t, \quad x_0^u = x \in L^2(\Lambda),$$

where $b(x) = x(x-1)(a-x)$.

Major changes:

- additive noise
- deterministic controls
- convex control domain
- nonlinearity not Lipschitz.

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$$J(u)=\mathbb{E}\left[\int_0^T\int_\Lambda \frac{c_{\overline{\Lambda}}}{2}\left(x_t^u(\lambda)-x_{\overline{\Lambda}}(t,\lambda)\right)^2+\frac{\nu}{2}u^2(t,\lambda)\mathrm{d}\lambda\mathrm{d}t+\int_\Lambda \frac{c_T}{2}\left(x_T^u(\lambda)-x^T(\lambda)\right)^2\mathrm{d}\lambda\right]$$

over $u \in L^2([0,T]\times\Lambda)$ subject to

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where $b(x) = x(x-1)(a-x)$.

## Theorem (Stannat, W., EECT 2021)

*The gradient of the cost functional is given by*

$$\nabla J(u)(t,\lambda) = \mathbb{E}\left[p_t(\lambda) + \nu u(t,\lambda)\right],$$

*where $p$ is the solution of the adjoint equation*

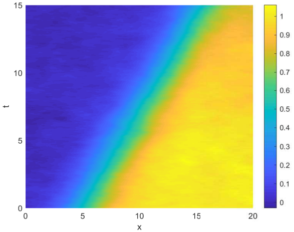
$$\begin{cases} -\partial_t p_t = \Delta p_t + b'(x_t^u)p_t + c_{\overline{\Lambda}}\left(x_t^u - x_{\overline{\Lambda}}(t,\cdot)\right), & t \in (0,T) \\ p_T = c_T\left(x_T^u - x^T\right) \in L^2(\Lambda). \end{cases}$$

# Simulations

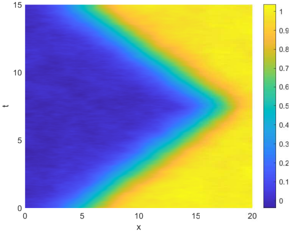
Consider

$$\begin{cases} \mathrm{d}x_t^u = [\Delta x_t^u + x_t^u(x_t^u - 1)(a - x_t^u) + u_t]\,\mathrm{d}t + \sigma \mathrm{d}W_t, & t \in [0, T] \\ x_0^u = x \in L^2(\Lambda), \end{cases}$$

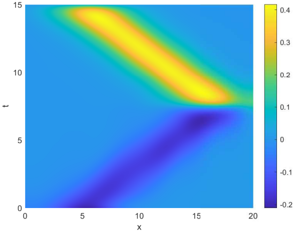
where $a = 39/40$, $\sigma = 1/2$, $T = 15$, $\Lambda = [0, 20]$.




Uncontrolled Solution




Controlled Solution




Optimal Control

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