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Nonlinear Feynman-Kac Formulae for B-Continuous Viscosity Solutions

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Setting

SPDE

$$\begin{cases} \mathrm{d}X_s^{t,x} = [AX_s^{t,x} + b(s, X_s^{t,x})]\mathrm{d}s + \sigma(s, X_s^{t,x})\mathrm{d}W_s, \quad s \in [t,T] \\ X_t^{t,x} = x \in H, \end{cases} \tag{1}$$

where

- $A : \mathcal{D}(A) \subset H \to H$ unbounded linear operator
- $b : [0,T] \times H \to H$ drift, $\sigma : [0,T] \times H \to L_2(\Xi, H)$ diffusion
- $(W_s)_{s\in[t,T]}$ cylindrical Wiener process on Ξ.

PDE

$$\begin{cases} v_t(t,x) + \langle Ax + b(t,x), Dv(t,x)\rangle_H + \frac{1}{2}\mathrm{tr}(\sigma^*(t,x)D^2v(t,x)\sigma(t,x)) \\ \qquad\qquad -f(t,x,v(t,x),\sigma^* Dv(t,x)) = 0, \quad (t,x) \in [0,T] \times H \\ v(T,x) = g(x), \quad x \in H, \end{cases} \tag{2}$$

where $f : [0,T] \times H \times \mathbb{R} \times \Xi \to \mathbb{R}$ and $g : H \to \mathbb{R}$.

BSDE

$$\begin{cases} \mathrm{d}Y_s^{t,x} = f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})\mathrm{d}s + Z_s^{t,x}\mathrm{d}W_s, \quad s \in [t,T] \\ Y_T^{t,x} = g(X_T^{t,x}). \end{cases} \tag{3}$$

Relationship under Smoothness Assumptions

Applying Itô's formula yields

$$\begin{aligned} &v(T, X_T^{t,x}) \\ &= v(s, X_s^{t,x}) + \int_s^T v_t(r, X_r^{t,x}) + \langle Dv(r, X_r^{t,x}), AX_r^{t,x} + b(r, X_r^{t,x})\rangle \mathrm{d}r \\ &\quad + \int_s^T \frac{1}{2}\mathrm{tr}(\sigma^*(r, X_r^{t,x})D^2v(r, X_r^{t,x})\sigma(r, X_r^{t,x}))\mathrm{d}r + \int_s^T \langle Dv(r, X_r^{t,x}), \sigma(r, X_r^{t,x})\mathrm{d}W_r\rangle \\ &= v(s, X_s^{t,x}) + \int_s^T f(r, X_r^{t,x}, v(r, X_r^{t,x}), \sigma^* Dv(r, X_r^{t,x}))\mathrm{d}r + \int_s^T \langle \sigma^* Dv(r, X_r^{t,x}), \mathrm{d}W_r\rangle \end{aligned}$$

Hence,

$$\begin{cases} Y_s^{t,x} = v(s, X_s^{t,x}) \\ Z_s^{t,x} = \sigma^* Dv(s, X_s^{t,x}) \end{cases}$$

solves BSDE (3). In particular, for $s = t$,

$$v(t,x) = Y_t^{t,x}. \tag{4}$$

Classical Feynman-Kac Formula

Consider special case $f(s,x,y,z) = k(s,x)y + l(s,x)$ for $k, l : [t,T] \times H \to \mathbb{R}$. Then (3) admits explicit solution

$$\begin{aligned} Y_s^{t,x} &= \exp\left(-\int_s^T k(r, X_r^{t,x})\mathrm{d}r\right) g(X_T^{t,x}) - \int_s^T \exp\left(-\int_s^r k(u, X_u^{t,x})\mathrm{d}u\right) l(r, X_r^{t,x})\mathrm{d}r \\ &\quad - \int_s^T \exp\left(-\int_s^r k(u, X_u^{t,x})\mathrm{d}u\right) Z_r^{t,x}\mathrm{d}W_r. \end{aligned}$$

Thus, taking the expectation in (4) yields

$$v(t,x) = \mathbb{E}\left[\exp\left(-\int_t^T k(r, X_r^{t,x})\mathrm{d}r\right) g(X_T^{t,x}) - \int_t^T \exp\left(-\int_t^r k(u, X_u^{t,x})\mathrm{d}u\right) l(r, X_r^{t,x})\mathrm{d}r\right].$$

Viscosity Solutions

Test Functions

A function ψ is a test function if $\psi = \varphi + h(t, \|x\|_H)$, where:

- $\varphi \in C^{1,2}((0,T) \times H)$ is locally bounded, B-lower semicontinuous, and φ_t, $A^*D\varphi$, $D\varphi$, $D^2\varphi$ are uniformly continuous on $(0,T) \times H$.
- $h \in C^{1,2}((0,T) \times \mathbb{R})$ and for every $t \in (0,T)$, $h(t,\cdot)$ is even and $h(t,\cdot)$ is non-decreasing on $[0, +\infty)$.

B-Continuous Viscosity Solutions

A locally bounded and upper semicontinuous function u on $(0,T] \times H$ which is B-upper semicontinuous on $(0,T) \times H$ is a viscosity subsolution of (2) if the following holds: whenever $u - \psi$ has a local maximum at a point $(t,x) \in (0,T) \times H$ for a test function ψ then

$$\begin{aligned} &\psi_t(t,x) + \langle x, A^*D\varphi(t,x)\rangle_H + \langle b(t,x), D\psi(t,x)\rangle_H \\ &\qquad + \frac{1}{2}\mathrm{tr}(\sigma^*(t,x)D^2\psi(t,x)\sigma(t,x)) - f(t,x,u(t,x),\sigma^*(t,x)D\psi(t,x)) \geq 0. \end{aligned}$$

Semilinear Feynman-Kac Formula (W. 2023+)

Main Assumptions

- A generates C_0-semigroup and satisfies strong B-condition
- b, σ linear growth
- b, f, g Lipschitz in x, y, z
- $\|\sigma(s,x) - \sigma(s,x')\|_{L_2(\Xi,H)} \leq C\|x - x'\|_{H_{-1}}$

Theorem

Let (Y, Z) be the solution of the BSDE (3) and define $u(t,x) := Y_t^{t,x}$. Then u is a B-continuous viscosity solution of the PDE (2).

Uniqueness

Additional Main Assumptions

- σ and f uniformly continuous on bounded sets
- f non-decreasing in v

Theorem

$u(t,x) := Y_t^{t,x}$ is the unique B-continuous viscosity solutions satisfying

$$\lim_{t\to T} |u(t,x) - g(S(T-t)x)| = 0$$

uniformly on bounded subsets of H and

$$|u(t,x)| \leq C_1 \exp\left(C_2 \left(\ln(1 + \|x\|_H)\right)^2\right)$$

for some constants $C_1, C_2 \geq 0$ and all $(t,x) \in (0,T) \times H$.

Nonlinear Feynman-Kac Formula

PDE

$$\begin{cases} v_t(t,x) + \langle Ax + b(t,x), Dv(t,x)\rangle_H + \frac{1}{2}\mathrm{tr}(\sigma^*(t,x)D^2v(t,x)\sigma(t,x)) \\ \qquad\qquad -f(t,x,v(t,x),Dv(t,x),D^2v(t,x)) = 0, \quad (t,x) \in [0,T] \times H \\ v(T,x) = g(x), \quad x \in H, \end{cases}$$

where $f : [0,T] \times H \times \mathbb{R} \times H \times \mathcal{S}(H) \to \mathbb{R}$.

Applying Itô's formula to $v(s, X_s^{t,x})$ and $Dv(s, X_s^{t,x})$ implies that

$$\begin{cases} Y_s^{t,x} = v(s, X_s^{t,x}) \\ Z_s^{t,x} = Dv(s, X_s^{t,x}) \end{cases} \qquad \begin{cases} \Gamma_s^{t,x} = D^2v(s, X_s^{t,x}) \\ \Lambda_s^{t,x} = \mathcal{L}_s Dv(s, X_s^{t,x}) \end{cases}$$

where $\mathcal{L}_s\varphi(s,x) = \partial_s\varphi(s,x) + \frac{1}{2}\mathrm{tr}(\sigma^*(s,x)D^2\varphi(s,x)\sigma(s,x))$, solves the 2BSDE

$$\begin{cases} \mathrm{d}Y_s^{t,x} = f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}, \Gamma_s^{t,x})\mathrm{d}s + \langle Z_s^{t,x}, \sigma(s, X_s^{t,x})\mathrm{d}W_s\rangle \\ \mathrm{d}Z_s^{t,x} = (\Lambda_s^{t,x} + \Gamma_s^{t,x}(AX_s^{t,x} + b(s, X_s^{t,x})))\mathrm{d}s + \Gamma_s^{t,x}\sigma(s, X_s^{t,x})\mathrm{d}W_s \\ Y_T^{t,x} = g(X_T^{t,x}). \end{cases}$$

References

Wessels, L. (2023+). Semilinear Feynman-Kac formulae for B-continuous viscosity solutions. *in preparation*.

Acknowledgements

The author acknowledges support of the German Academic Exchange Service (DAAD) via a postoctoral fellowship.

DAAD Deutscher Akademischer Austauschdienst German Academic Exchange Service